

Spin textures in quantum Hall systems

Benoît Douçot

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1 Introduction

The subject of these lectures combines several different manifestations of topology in a condensed matter system. The most classical one is through the notion of texture. By this, we mean any non-singular and topologically non-trivial spatial configuration of some relevant order parameter. Textures are therefore qualitatively different from defects, for which the order parameter field exhibits a point-like singularity in $2D$ space, or more generally a codimension 2 surface of singularity in D -dimensional space. In this latter situation, the set of points where the order parameter field is smooth exhibits a non-trivial topology, equivalent to the one of a circle S^1 . Denoting by M the order parameter manifold, defects are naturally classified by the group $\pi_1(M)$ of homotopy classes of smooth maps from S^1 to M [1, 2, 3]. By contrast, textures with a finite energy correspond to configurations in which the order parameter is uniform at infinity, which allows us to compactify physical space into a D -dimensional sphere S^D . Textures are then classified according to the higher homotopy group $\pi_D(M)$. In most systems, textures appear as finite energy excitations above an ordered ground-state. A remarkable aspect of quantum Hall ferromagnets is that non trivial textures have been predicted to form, if the electronic g factor is not too large, as soon as electrons are added to or removed from a filled Landau level [4]. Spin textures on a $2D$ system are classified by $\pi_2(S^2) = \mathbb{Z}$, so they carry an integer topological charge N_{top} . A striking prediction of Sondhi et al. is that N_{top} is identical to the electric charge: it is equal to $+1$ for a hole (Skyrmion) and to -1 for an electron (anti-Skyrmion) [4]. This picture has been confirmed experimentally, in particular thanks to NMR measurements of the electronic spin susceptibility [5] and nuclear spin relaxation [6]. Experimentally, it is easier to control the Skyrmion density $1 - \nu$ than their total number. Here ν denotes, as usual, the filling factor of the lowest Landau level. For a small but finite Skyrmion density, it has been predicted that the long range Coulomb interaction between the charges bound to Skyrmions will favor their ordering into a $2D$ periodic lattice [7, 8]. Several experiments have provided substantial evidence for the existence of Skyrmion lattices in $2D$ electron gases close to $\nu = 1$. Let us mention for example specific heat measurements [9, 10], NMR relaxation [11], Raman spectroscopy [12], and microwave pinning-mode resonances [13].

More recently, the physics of quantum Hall ferromagnets has been stimulated by the discovery of new systems, which can provide access to more than two internal states for each electron. The first of these has been the quantum Hall bilayer [14], in which, besides the physical electronic spin, the additional bilayer degree of freedom can be viewed as a

kind of isospin. Skyrmions in these systems have been studied in great detail [15]. Unfortunately, bilayers are far from the maximal $SU(4)$ symmetry that one may expect in a system with four possible internal states. The discovery of graphene opened a very promising way to achieve such a large symmetry. In graphene, the isospin degree of freedom is implemented thanks to the existence of two inequivalent Dirac points. In the presence of an external magnetic field B , it has been shown that the couplings which break $SU(4)$ symmetry are smaller than the symmetry preserving ones by a factor a/l , where a is the lattice spacing and $l = \sqrt{\hbar/eB}$ is the magnetic length [16, 17]. Other possible examples of systems with more than two internal states are semi-conductors with valley degeneracy [18, 19, 20], and cold atoms [21, 22]. Theoretical works have been dedicated to the elucidation of phase diagrams for skyrmionic matter in the presence of various physically relevant interactions and anisotropies [23, 21, 22], and the computation of the associated collective mode spectrum [24]. These later calculations have been partly motivated by NMR relaxation rate measurements on bilayer systems [25, 26]. Recently, we have revisited these questions for fermions with d internal states and for $SU(d)$ -symmetric effective Hamiltonians [27]. This high symmetry allowed us to set up an accurate variational calculation for the optimal wave-function describing a periodic lattice of Skyrmions, for which a simple analytic expression has been obtained. Because these periodic states fully break the underlying $SU(d)$ symmetry, we expect a collective mode spectrum composed of $d^2 - 1$ Goldstone branches and one magnetophonon branch. These expectations have been confirmed by explicit calculations based on a time-dependent Hartree-Fock treatment of our $SU(d)$ -symmetric effective Hamiltonian.

The goal of these lectures is to provide a theory-oriented introduction to the physics of textures in quantum Hall ferromagnets, so they do not attempt to review this already rich subject, and many important aspects will not be mentioned. To give an idea, the APS web-site records 500 citations for the paper by Sondhi et al [4]. Our recent approach on periodic textures will be presented in section 3, and the associated collective modes will be the subject of section 4. But before discussing our contributions, I have tried to show in some detail how to derive the effective models which we use from microscopic models of interacting fermions in the lowest Landau level. Establishing this connection is the goal of section 2. Most of the results there are already quite old, and due to many researchers [4, 15, 28, 29, 30, 31]. I have tried to give a unified presentation of these seminal works using the framework of coherent state quantization [32, 33]. This formalism appears at two stages, with different manifestations and purposes. The first one is to associate a Slater determinant $|\mathcal{S}_\psi\rangle$ composed of single electron orbitals in the lowest Landau level to a prescribed texture, described in terms of a smooth d component spinor field $\psi_a(r)$, ($1 \leq a \leq d$). Coherent state quantization is used to construct precisely $|\mathcal{S}_\psi\rangle$ and to compute the expectation values of some physical observables such as the particle density or the interaction energy. The key remark here is that projection onto the lowest Landau level turns the physical plane into a two-dimensional phase-space, in which each single particle quantum state occupies an area equal to $2\pi l^2$. In the strong field limit, this area goes to zero as $1/B$, so we have a kind of classical limit, in which we can neglect the non commutation between the two guiding center coordinates $\hat{R}_x$ and $\hat{R}_y$. Going away from this limit yields naturally a gradient expansion in which the small parameter is nl^2 , where n is the average topological charge density. The second use of coherent state quantization is at the many-particle level. We can indeed view the Slater determinants $|\mathcal{S}_\psi\rangle$ as coherent states for the many fermion problem, which span a low-

energy subspace within the fermionic Fock space. Although the mathematical aspects here are not as clear to us as for the single particle level, this viewpoint can still provide a useful intuition for the many-body problem [34, 35], and will motivate our subsequent treatment of collective dynamics in section 4.

2 Physical properties of spin textures

2.1 Intuitive picture

Before beginning our discussion of textures in quantum Hall systems, it is useful to remind a few aspects of the physics of a single charge e particle moving on a $2D$ plane in a strong uniform magnetic field. In this limit, the particle undergoes a fast cyclotron motion, with characteristic frequency $\omega_c = eB/m$. In the absence of disorder, the corresponding energy spectrum is quantized according to $E_n = \hbar\omega_c(n+1/2)$ with n a non-negative integer. Each of these Landau levels is infinitely degenerate in the thermodynamic limit. The physical origin of this degeneracy can be understood in classical terms: in the absence of disorder, the center of cyclotron orbits can be located anywhere on the plane, and all these locations give the same energy. The coordinates R_x and R_y of this so-called guiding center are:

$$R_x = \frac{x}{2} + \frac{p_y}{eB} \quad R_y = \frac{y}{2} - \frac{p_x}{eB} \quad (1)$$

Quantum mechanically, they become operators, $\hat{R}_x$ and $\hat{R}_y$, whose commutator is:

$$[\hat{R}_x, \hat{R}_y] = -il^2 \quad (2)$$

In the limit of a very strong magnetic field, the cyclotron gap $\hbar\omega_c$ is larger than other energy scales, in particular the scale e^2/l associated to Coulomb interactions, so we may project all the single particle states onto the highest occupied Landau level. To simplify the discussion, we shall assume that the electronic filling factor ν is less than d , so this Landau level is the lowest one, corresponding to $n = 0$ and often denoted by the LLL acronym. Physically, the only remaining degrees of freedom for a given electron are, besides its d internal levels, its guiding center coordinates $\hat{R}_x$ and $\hat{R}_y$. Eq. (2) shows that they behave as a canonically conjugate pair of observables. This turns the physical plane into a $2D$ phase-space, with an effective Planck's constant $\hbar_{\text{eff}} = l^2$. Heisenberg's uncertainty principle suggests then that each quantum state in the LLL occupies an area $2\pi l^2$, so the degeneracy of the LLL should be equal to the system area divided by $2\pi l^2$, which is equal to the total magnetic flux through the system divided by the flux quantum $\Phi_0 = h/e$: we may also say that each single particle state in the LLL occupies the area corresponding to one flux quantum. This intuitive estimate turns out to give the exact degeneracy for each Landau level.

On physical ground, the fact that a single charge added to or removed from a fully polarized state at $\nu = 1$ binds a spin texture is relatively easy to understand. Let us, to be specific, remove one electron. The remaining $N - 1$ electrons would lower their electrostatic energy if they could benefit of the created hole to move slightly away from each other. However, as we have just discussed, the physical plane becomes like a phase space, with each single particle state occupying an area threaded by one flux quantum. The natural way to create a smooth distribution of the remaining $N - 1$ electrons would

be to remove one flux quantum through the whole plane. In the absence of a Zeeman spin anisotropy, this can be done with a small energy cost by twisting the spins slowly. The Berry phase associated to the spin texture creates an artificial effective magnetic field, which adds to the physical external field.

To formulate the previous remarks in more precise language, we describe the spin texture by a smooth d -component spinor field $\psi_a(r)$ ($1 \leq a \leq d$), which is expected to have only small variations on the scale of the magnetic length. In this presentation, I will often denote this spinor field by $|\psi(r)\rangle$. This notation is suggestive and convenient, although it has a potential risk of confusion, even at the single particle level, by the presence of the r variable inside the ket. I trust the reader not to be misled by this choice of notation. We suppose that the quantum state of the $N - 1$ electron system can be described by a Slater determinant obtained from a collection of single particle orbitals $|\Phi_\alpha\rangle$ ($1 \leq \alpha \leq N - 1$) which have the form:

$$\Phi_{\alpha,a}(r) = \chi_\alpha(r)\psi_a(r) \quad (3)$$

where $\chi_\alpha(r)$ describes the remaining orbital degree of freedom, once the spins are constrained to follow the prescribed texture $\psi_a(r)$. We wish first to minimize the kinetic energy of this state. A good guide is to minimize separately the kinetic energy of the individual orbitals $|\Phi_\alpha\rangle$. A simple calculation shows that:

$$\langle \Phi_\alpha | (P - eA)^2 | \Phi_\alpha \rangle = \langle \chi_\alpha | (P - eA_{\text{eff}})^2 + V_{\text{eff}} | \chi_\alpha \rangle \quad (4)$$

with:

$$A_{\text{eff}} = A - \frac{\Phi_0}{2\pi} \mathcal{A}, \quad \mathcal{A} = \frac{1}{i} \langle \psi | \nabla | \psi \rangle \quad (5)$$

and:

$$V_{\text{eff}} = \langle \nabla \psi | \nabla \psi \rangle - \langle \nabla \psi | \psi \rangle \langle \psi | \nabla \psi \rangle \quad (6)$$

$\mathcal{A}$ is often referred to as the *Berry connection* associated to the spin texture $|\psi(r)\rangle$. The above expressions (5) and (6) are valid provided the spinor field $|\psi(r)\rangle$ is everywhere normalized to unity, that is we impose $\langle \psi(r) | \psi(r) \rangle = 1$ for all r . The total kinetic energy is minimized by putting all the effective orbitals $\chi_\alpha(r)$ in the lowest Landau level of the effective magnetic field corresponding to A_{eff} , in the presence of the potential V_{eff} . Because V_{eff} is small for a slowly varying texture, we expect that its effect is subleading compared to A_{eff} . This level contains N_{eff} states, where N_{eff} is simply equal to the number of flux quantas of the effective magnetic field through the system. Now it is an important fact that the total flux associated to the Berry connection is 2π times an integer N_{top} called the *total topological charge*. To be more precise, this sharp "quantization" holds when the spin texture goes to a constant value $|\psi(\infty)\rangle$ far from the origin in all directions. This allows us to compactify the plane, which is topologically trivial (it can be shrunk smoothly into a point), into a two-dimensional sphere S^2 , which has a non-trivial topology.

At this point, it is important to note that there is clearly a gauge freedom in the definition of the spinor field $|\psi(r)\rangle$. As shown by eq.(3), a change of $|\psi(r)\rangle$ into $\exp(i\theta(r))|\psi(r)\rangle$, where the phase $\theta(r)$ is an arbitrary function, can be compensated by the related change of $\chi_\alpha(r)$ into $\exp(-i\theta(r))\chi_\alpha(r)$, so that the physical orbitals $|\Phi_\alpha\rangle$ remain unchanged. It is then more appropriate to view $|\psi(r)\rangle$ as a representative of the complex line it generates in the complex space $\mathbb{C}^d$. In mathematical terms, the spinor field $|\psi(r)\rangle$ should be viewed as a map from S^2 to the complex projective space $\mathbb{C}P(d - 1)$. This map can be used to

define a line bundle over S^2 , whose Chern number is the topological charge N_{top} . Coming back to the main discussion, we have therefore the very important relation:

$$N_{\text{eff}} = N - N_{\text{top}} \quad (7)$$

If we wish to describe a system with one hole, we need to get $N_{\text{eff}} = N - 1$, so that $N_{\text{top}} = 1$. Likewise $N_{\text{top}} = -1$ for an added electron. This shows that topologically non-trivial textures (called Skyrmions) are bound to extra charges, the driving force being the Coulomb interaction.

Before going further, let us give an illustration of a single Skyrmion texture in the $d = 2$ case. Then, we can associate to each spinor (ψ_1, ψ_2) the expectation value $\mathbf{n}(r) = \langle \psi(r) | \boldsymbol{\sigma} | \psi(r) \rangle$, where $\boldsymbol{\sigma} = (\sigma_x, -\sigma_y, \sigma_z)$ denotes the usual Pauli spin matrices. We can use this map from $\mathbb{CP}(1)$ to S^2 to construct a spin texture if we choose a map from the physical plane to $\mathbb{CP}(1)$. For $r = (x, y)$ in the plane, let us pick $\psi_1 = z$ and $\psi_2 = 1$, where $z = x + iy$. At the origin $z = 0$, $\mathbf{n}(0) = -\mathbf{e}_z$, and far from it, $\mathbf{n}(\infty) \rightarrow \mathbf{e}_z$. Explicitly, we get:

$$\mathbf{n}(x, y) = \left(\frac{2x}{|r|^2 + 1}, \frac{2y}{|r|^2 + 1}, \frac{|r|^2 - 1}{|r|^2 + 1} \right), \quad |r|^2 = x^2 + y^2 \quad (8)$$

This map from the plane to S^2 is nothing but the inverse of the stereographic projection of S^2 to the $z = 0$ plane from the north pole. To find $\mathbf{n}(x, y)$ we simply draw the line joining the point $(x, y, 0)$ to the north pole $(0, 0, 1)$. This line intersects the unit sphere $x^2 + y^2 + z^2 = 1$ at a second point, besides the north pole, which is simply $\mathbf{n}(x, y)$. This construction is depicted on Fig. 1. It shows clearly that each spin value on S^2 (with the exception of the north pole) is reached exactly once under the inverse stereographic projection. We also note that when $|r| = 1$, $\mathbf{n}(x, y) = (x, y, 0)$, so the spin configuration, restricted to the unit circle, coincides with a 2π vortex. However, unlike the vortex, which has a singular core and no well defined limit as $|r| \rightarrow \infty$, the texture is everywhere smooth and reaches the north pole if $|r| \rightarrow \infty$ along all possible directions. This qualitative difference with a 2π vortex is obtained by allowing spins to move away from the $z = 0$ plane.

Although physically appealing, the previous discussion is not totally satisfactory. Its main problem is that minimizing the expectation value $\langle \chi_\alpha | (P - eA_{\text{eff}})^2 + V_{\text{eff}} | \chi_\alpha \rangle$ doesn't imply that the single particle states $|\Phi_\alpha\rangle$ belong to the lowest Landau level (corresponding to the *physical* magnetic field). This problem disappears if one considers, as V. Pasquier [29, 30], the case of a positive topological charge and a subset of textures in which the components $\psi_a(r)$ of $|\psi(r)\rangle$ are *analytic* functions of $z = x + iy$. Because the lowest Landau level corresponds (in the circular gauge) to wave functions of the form $\psi(r) = f(z) \exp(-|z|^2/(4l^2))$ with $f(z)$ analytic, we see that if the orbital parts $\chi_\alpha(r)$ are of this form, multiplying them by an analytic spinor $\psi_a(r)$ will produce single particle states $\Phi_{\alpha,a}(r)$ which are also in the lowest Landau level. As shown by MacDonald, Fertig and Brey [28], Slater determinants associated to such textures are exact ground-states for a model with $N_{\text{el}} < N$ electrons and a point-like interaction. However, it is not always sufficient to restrict ourselves to analytic textures. The first reason is that in the presence of spin anisotropies, or even isotropic but long range interactions, the optimal textures are no longer analytic. Another reason is that we wish to keep the possibility to study excited states, which live outside the analytic subspace. In spite of these restrictions, these works by V. Pasquier have brought important ideas, in particular the realization that we

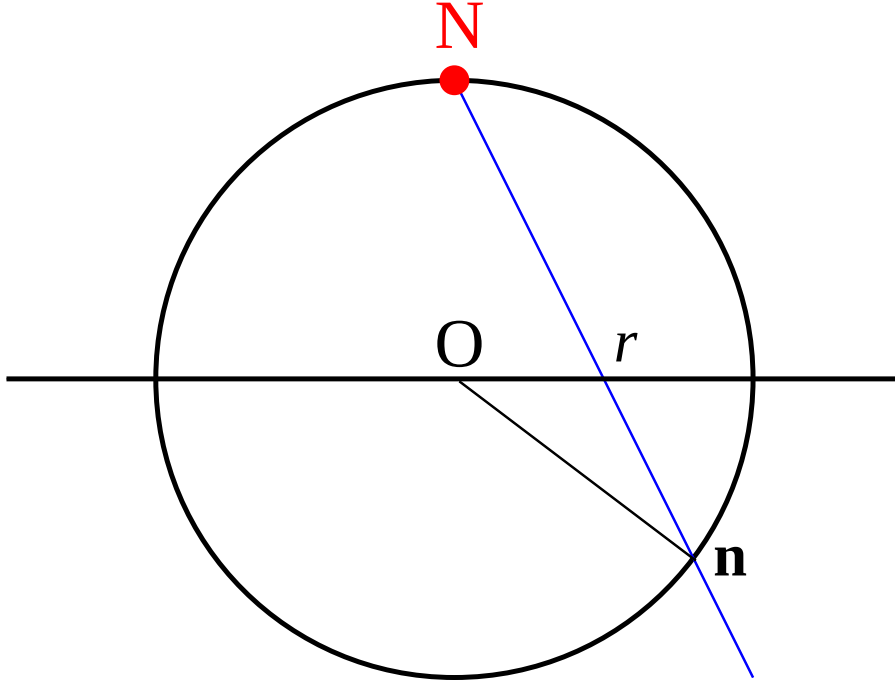


Figure 1: Stereographic projection used to construct a single Skyrmion. To find the spin orientation $\mathbf{n}$ at a point r on the $z = 0$ plane, one has simply to find the intersection between the unit sphere and the line joining r to the north pole N .

can use tools from geometric quantization to analyze physical properties of quantum Hall textures. In this lectures, I will show that geometric quantization is also a very useful tool to construct a large class of textures, not limited to analytic functions.

2.2 Construction of spin textures

Guided by the previous discussion, we would like to find a way to associate a Slater determinant $|\mathcal{S}_\psi\rangle$ for N_e electrons in the lowest Landau level, to a smooth spinor field $\psi_a(r)$ ($1 \leq a \leq d$). On physical ground, this could be achieved by switching on the following auxiliary Zeeman-like Hamiltonian acting on a single particle wave-function $\phi_a(r)$:

$$(\hat{H}_{\psi,\text{cl}}\phi)_a(r) = -\psi_a(r) \sum_{b=1}^d \psi_b^*(r) \phi_b(r) \quad (9)$$

The ground-state of $\hat{H}_{\psi,\text{cl}}$ is infinitely degenerate, being composed of all the single particle states of the form $\phi_a(r) = f(r)\psi_a(r)$, where $f(r)$ is an arbitrary function with complex values. Excited eigenstates of $\hat{H}_{\psi,\text{cl}}$ are also infinitely degenerate, since they correspond to wave-functions $\phi_a(r)$ which are orthogonal at every point r to the local spinor $\psi_a(r)$, that is $\sum_{b=1}^d \psi_b^*(r) \phi_b(r) = 0$ everywhere. In the following discussion, it will be convenient to view single particle Hamiltonians as $d \times d$ matrices whose entries are operators acting on the orbital part of the wave-function. With this convention, we may write $\hat{H}_{\psi,\text{cl}}$ as:

$$(\hat{H}_{\psi,\text{cl}})_{ab} = -\psi_a(r)\psi_b^*(r), \quad 1 \leq a, b \leq d \quad (10)$$

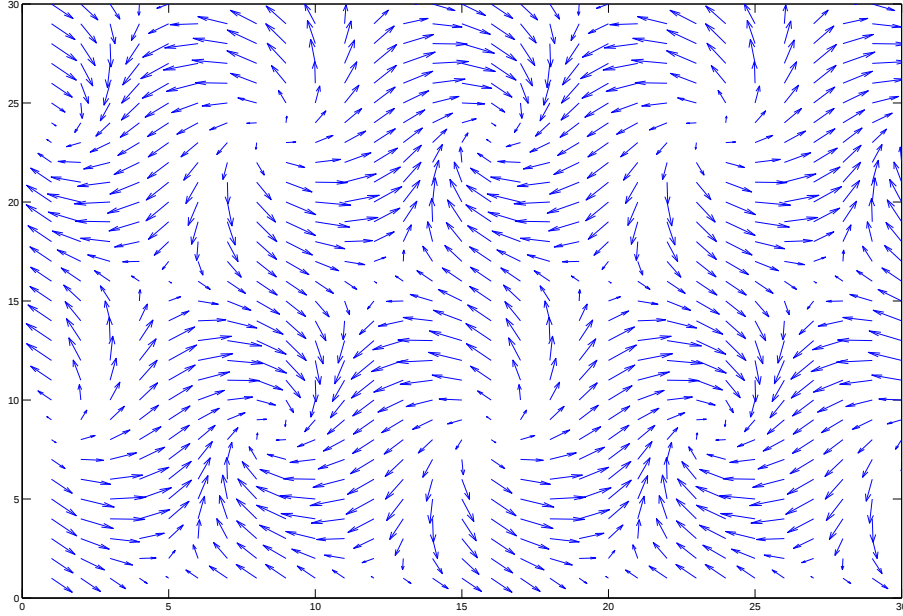


Figure 2: A periodic spin texture for the $SU(2)$ case ($d = 2$). The projection of the spins on the horizontal plane is shown. This triangular pattern corresponds to the optimal variational state described in section 3.4.

It turns out that this is not quite the construction that we need, because it doesn't take into account the projection on the lowest Landau level. Denoting by $\mathcal{P}_{\text{LLL}}$ the self adjoint projector onto this level, a natural candidate to create a texture would be to look at the ground-state of $\hat{H}_\psi = \mathcal{P}_{\text{LLL}} \hat{H}_{\psi, \text{cl}} \mathcal{P}_{\text{LLL}}$, or with the previous notation:

$$(\hat{H}_\psi)_{ab} = -\mathcal{P}_{\text{LLL}} \psi_a(r) \psi_b^*(r) \mathcal{P}_{\text{LLL}}, \quad 1 \leq a, b \leq d \quad (11)$$

As we explained in section 2.1, the projection on the lowest Landau level turns the physical plane into a two-dimensional phase-space, in which coordinates R_x and R_y become canonically conjugated. In this quantization process, the role of Planck's constant $\hbar$ is played by the square of the magnetic length l , in tune with the general picture that each quantum state occupies an area $2\pi\hbar$ in phase-space. The limit of large magnetic fields, where l is much smaller than the characteristic length associated to the spatial variations of the texture field $\psi_a(r)$, can therefore be viewed as a *semi-classical limit*. In this limit, we expect the following qualitative properties for the spectrum of $\hat{H}_\psi$ [36]: the two degenerate levels with eigenvalues -1 and 0 are replaced by two bands whose widths are at most proportional to l^2 , containing $N - N_{\text{top}}$ and $(d - 1)N + N_{\text{top}}$ states respectively, and separated by a well defined gap. It is then natural to define the quantum state associated to the smooth texture $\psi_a(r)$ as the Slater determinant $|\mathcal{S}_\psi\rangle$ composed of the $N - N_{\text{top}}$ single particle states lying in the lowest band of $\hat{H}_\psi$. The goal of this section is to investigate the physical properties of such states. With this purpose in mind, all the relevant information is encoded in the projector $\hat{P}_\psi$ on this lowest band. Our first task is then to write explicit expressions for $\hat{P}_\psi$ in terms of the texture $\psi_a(r)$.

The two equations that we wish to solve are:

$$[\hat{H}_\psi, \hat{P}_\psi] = 0 \quad (12)$$

$$\hat{P}_\psi \hat{P}_\psi = \hat{P}_\psi \quad (13)$$

The main difficulty here is that we wish to diagonalize a $d \times d$ matrix $(\hat{H}_\psi)_{ab}$ whose elements are themselves operators, instead of numbers. A very important remark is that these operators do commute in the classical limit $l^2 \rightarrow 0$, so the usual diagonalization methods (designed to work with commuting numbers) can be applied there. One can hope then that the non-commutating nature of the elements $(\hat{H}_\psi)_{ab}$ can be dealt with in a semi-classical expansion. That this is indeed possible is well known in the mathematical litterature, where it has been shown that the projector $\hat{P}_\psi$ can be constructed as a formal power series in l^2 [37, 38].

In doing these calculations, it will be useful to make use of the correspondence principle between classical and quantum mechanics, to represent operators such as $(\hat{H}_\psi)_{ab}$ and $(\hat{P}_\psi)_{ab}$ in terms of functions of the underlying phase-space coordinates x and y . It is a general fact that there are many ways to represent operators by functions over classical phase-space in such a way that standard classical mechanics emerges as a limit of this quantization process when $\hbar \equiv l^2 \rightarrow 0$. Rather than being a problem, this ambiguity has positive aspects, because it allows us to choose the precise correspondence which simplifies our calculations. In the present problem, we shall use the so-called covariant symbol in Berezin's terminology [33], also called the Husimi distribution in the quantum optics and quantum chaos communities.

For a given operator $\hat{f}$ acting in the lowest Landau level, the associated covariant symbol $f(z, \bar{z})$ is simply the expectation value of $\hat{f}$ taken on the normalized coherent state $|\Phi_{\bar{z}}\rangle$ centered at (x, y) with $z = x + iy$. To bring more fluidity to this presentation, the precise definition and elementary properties of these coherent states are relegated to Appendix 5.1. There, we also show that a given operator is uniquely determined by its covariant symbol, from which it can be explicitly constructed through the normal ordering procedure.

The non-commuting algebra of quantum operators can be represented as a deformation of the commuting algebra of functions over phase-space. To see this, we need to know the covariant symbol of the product of two operators. It is common to write $\widehat{\hat{f}\hat{g}} = \widehat{f \star g}$, where the explicit formula for the star product is:

$$f \star g = f \exp \left(-i \frac{l^2}{2} (\overleftarrow{\partial}_x \overrightarrow{\partial}_y - \overleftarrow{\partial}_y \overrightarrow{\partial}_x) + \frac{l^2}{2} (\overleftarrow{\partial}_x \overrightarrow{\partial}_x + \overleftarrow{\partial}_y \overrightarrow{\partial}_y) \right) g \quad (14)$$

This may be represented as a power series in l^2 :

$$f \star g = \sum_{n=0}^{\infty} l^{2n} f \star_n g \quad (15)$$

with:

$$f \star_0 g = fg \quad (16)$$

$$f \star_1 g = -\frac{i}{2} \{f, g\} + \frac{1}{2} \nabla f \cdot \nabla g \quad (17)$$

The Poisson bracket is defined as usual by:

$$\{f, g\} = \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \quad (18)$$

Equations (16) and (17) imply that:

$$[\hat{f}, \hat{g}] = -il^2 \widehat{\{f, g\}} + \mathcal{O}(l^4), \quad (19)$$

which is an expression of the correspondence principle between classical and quantum mechanics.

With this choice of correspondence between functions and operators, it is natural to modify accordingly the definition of the single particle Hamiltonian $\hat{H}_\psi$ and to replace eq. (11) by:

$$(\hat{H}_\psi)_{ab} = -\psi_a(r) \widehat{\psi_b^*(r)}, \quad 1 \leq a, b \leq d \quad (20)$$

This new definition does not have any effect on the physics, because it simply modifies the way we parametrize Slater determinants $|\mathcal{S}_\psi\rangle$ in terms of classical spinor fields $\psi_a(r)$.

Let us write then $\hat{P}_\psi = \hat{P}_0 + l^2 \hat{P}_1 + \mathcal{O}(l^4)$, and compute the first two terms $\hat{P}_0$ and $\hat{P}_1$ in the semi-classical expansion of $\hat{P}_\psi$. We have dropped the ψ subscript in $\hat{P}_0$ and $\hat{P}_1$ to lighten the notation. From now on we shall replace operators like $\hat{H}_\psi$ and $\hat{P}_\psi$ by their $d \times d$ matrix symbols H_ψ and P_ψ . For two such matrix symbols $A \equiv A_{ij}(r)$ and $B \equiv B_{ij}(r)$, the matrix star product $A \star B$ is defined by the usual matrix multiplication rule, in which the ordinary product is replaced by the star product, that is :

$$(A \star B)_{ik} = \sum_j A_{ij} \star B_{jk} \quad (21)$$

Similarly, we define:

$$[A, B]_\star = A \star B - B \star A \quad (22)$$

To zeroth order in l^2 , eqs. (12) and (13) give:

$$[H_\psi, P_0] = 0 \quad (23)$$

$$P_0 P_0 = P_0 \quad (24)$$

This is the standard diagonalization problem for an hermitian matrix. Because we are interested in the eigenvalue of $\hat{H}_\psi$ which goes to -1 in the classical limit, we choose:

$$(P_0)_{ij} = \psi_i \psi_j^* \quad (25)$$

Here, we assumed that the local spinor field is everywhere normalized, that is $\sum_j |\psi_j(r)|^2 = 1$ for any r .

The first order terms in eqs. (12) and (13) read:

$$[H_\psi, P_1] + [H_\psi, P_0]_1 = 0 \quad (26)$$

$$P_0 P_1 + P_1 P_0 + P_0 \star_1 P_0 = P_1 \quad (27)$$

Here, we used the notation:

$$[A, B]_\star = \sum_{n=0}^{\infty} l^{2n} [A, B]_n \quad (28)$$

These equations can be seen as ordinary matrix equations for P_1 :

$$[H_\psi, P_1] = -[H_\psi, P_0]_1 \quad (29)$$

$$P_1 - P_0 P_1 - P_1 P_0 = P_0 \star_1 P_0 \quad (30)$$

Because $P_0 = -H_\psi$, $[H_\psi, P_0]_\star = 0$ which implies $[H_\psi, P_0]_1 = 0$. Therefore P_1 has to commute with P_0 . Multiplying eq. (30) by P_0 on both sides and subtracting the two results gives a necessary condition for the existence of P_1 :

$$P_0(P_0 \star_1 P_0) = (P_0 \star_1 P_0)P_0 \quad (31)$$

This condition is always satisfied, because the star product is associative: starting from $P_0 \star (P_0 \star P_0) = (P_0 \star P_0) \star P_0$ and keeping the first order term in l^2 gives exactly eq. (31). Using this property, it is easy to find that P_1 is given by:

$$P_1 = (1 - 2P_0)(P_0 \star_1 P_0) \quad (32)$$

This is the most important result of this section. We emphasize that it holds for an arbitrary smooth spinor field $\psi_a(r)$. The small parameter in the expansion of P_ψ is just the ratio between the typical length over which $\psi_a(r)$ varies and the magnetic length l . Let us check that this result allows us to compute the local charge density bound to the texture. From (32) and (17), using the classical expression (25) for P_0 gives:

$$\begin{aligned} (P_1)_{jk} = & -\frac{i}{2} (\{\psi_j, \psi_k^*\} + \{\psi_j, \psi_l\} \psi_l^* \psi_k^* + \psi_j \psi_l \{\psi_l^*, \psi_k^*\} + \psi_j \{\psi_l, \psi_l^*\} \psi_k^*) \\ & + \frac{1}{2} \nabla \psi_j \cdot \nabla \psi_k^* + \frac{1}{2} (\nabla \psi_j \cdot \nabla \psi_l) \psi_l^* \psi_k^* + \frac{1}{2} \psi_j \psi_l (\nabla \psi_l^* \cdot \nabla \psi_k^*) \\ & - \frac{1}{2} \psi_j (\nabla \psi_l \cdot \nabla \psi_l^* - 2\psi_l (\nabla \psi_m \cdot \nabla \psi_m^*) \psi_m^*) \psi_k^* \end{aligned} \quad (33)$$

It can be checked that this expression is invariant under local gauge transformations. A direct consequence is:

$$\text{Tr}(P_1(r)) = -\mathcal{B}(r) = -2\pi Q(r) \quad (34)$$

Here, $\mathcal{B} = \partial_x \mathcal{A}_y - \partial_y \mathcal{A}_x$ is the curvature of the Berry connection, and $Q(r)$ is the topological charge density associated to the ψ texture. The local particle density $\rho(r) = \sum_a \langle \psi_a^+(r) \psi_a(r) \rangle$ is equal to $(1/2\pi l^2) \text{Tr}(P_\psi(r))$, as can be seen from eq. (132) in Appendix 5.3. Therefore:

$$\rho(r) = \frac{1}{2\pi l^2} - Q(r) + \mathcal{O}(l^2) \quad (35)$$

This is a local form of eq. (7), because $1/2\pi l^2 = B/\Phi_0$ is the particle density in a filled Landau level. Because the integrated result, eq. (7), is a relation between integers, we expect that the spatial integral of all higher order corrections in l^2 to $\rho(r)$ are equal to zero.

2.3 Energetics of spin textures

We consider the usual $SU(d)$ -symmetric Hamiltonian with two-body potential $V(r - r')$:

$$H = \frac{1}{2} \sum_{ab} \int d^2r \int d^2r' V(r - r') \Psi_a^+(r) \Psi_b^+(r') \Psi_b(r') \Psi_a(r) \quad (36)$$

where the single particle creation and annihilation operators are projected onto the lowest Landau level, as detailed in subsection 5.3. In the Slater determinant associated to the classical texture $\psi_a(r)$, we can use Wick's theorem to evaluate the expectation value of H , and we get $\langle H \rangle_\psi = \langle H \rangle_{H,\psi} + \langle H \rangle_{F,\psi}$ with:

$$\langle H \rangle_{H,\psi} = \frac{1}{2} \sum_{ab} \int d^2r \int d^2r' V(r-r') \langle \Psi_a^+(r) \Psi_a(r) \rangle_\psi \langle \Psi_b^+(r') \Psi_b(r') \rangle_\psi \quad (37)$$

and:

$$\langle H \rangle_{F,\psi} = -\frac{1}{2} \sum_{ab} \int d^2r \int d^2r' V(r-r') \langle \Psi_a^+(r) \Psi_b(r') \rangle_\psi \langle \Psi_b^+(r') \Psi_a(r) \rangle_\psi \quad (38)$$

Using eq. (134) in Appendix 5.3, we can express these energies in terms of the matrix symbol $P_\psi(r)$ as:

$$\langle H \rangle_{H,\psi} = \frac{1}{2(2\pi l^2)^2} \int d^2r \int d^2r' V(r-r') \text{Tr}(P_\psi(r)) \text{Tr}(P_\psi(r')) \quad (39)$$

and:

$$\langle H \rangle_{F,\psi} = -\frac{1}{2(2\pi l^2)^2} \int d^2r \int d^2r' V(r-r') \text{Tr}(P_\psi(s(r,r')) P_\psi(s(r',r))) \exp\left(-\frac{(r-r')^2}{2l^2}\right) \quad (40)$$

Here, we have set:

$$s(r, r') = \frac{r+r'}{2} + \frac{i}{2} \hat{z} \times (r' - r) \quad (41)$$

Let us first consider the case of a point-like interaction, $V(r-r') = W\delta(r-r')$. We get:

$$\langle H \rangle_\psi = \frac{W}{2(2\pi l^2)^2} \int d^2r [(\text{Tr}(P_\psi(r)))^2 - \text{Tr}(P_\psi(r)^2)] \quad (42)$$

For a fully polarized system, we can choose a basis in internal space such that $(P_\psi)_{ab}(r) = \delta_{a1}\delta_{b1}f(r)$, and $\langle H \rangle_\psi = 0$ as expected, because the orbital wave function is completely antisymmetric, so it is impossible for two particles to be at the same point. Expanding the integrand in eq. (42), we get:

$$(\text{Tr}(P_\psi(r)))^2 - \text{Tr}(P_\psi(r)^2) = 2l^2(\text{Tr}P_1 - \text{Tr}(P_0P_1)) + \mathcal{O}(l^4) \quad (43)$$

From eq. (33), we find:

$$\text{Tr}(P_0P_1) = -\frac{\mathcal{B}}{2} - \frac{1}{2} (\langle \nabla\psi | \nabla\psi \rangle - \langle \nabla\psi | \psi \rangle \langle \psi | \nabla\psi \rangle) \quad (44)$$

Putting everything together gives:

$$\langle H \rangle_\psi = -\frac{W}{4\pi l^2} N_{\text{top}} + \frac{W}{8\pi^2 l^2} \int d^2r (\langle \nabla\psi | \nabla\psi \rangle - \langle \nabla\psi | \psi \rangle \langle \psi | \nabla\psi \rangle) \quad (45)$$

We now turn to the physically important case of Coulomb interaction, $V(r) = e^2/4\pi\epsilon r$. In fact, the Hartree contribution is easily written for an arbitrary interaction. Eq. (37) becomes, taking into account (35):

$$\langle H \rangle_{H,\psi} = \frac{1}{2} \int d^2r \int d^2r' V(r-r') \left(\frac{1}{2\pi l^2} - Q(r) + \mathcal{O}(l^2) \right) \left(\frac{1}{2\pi l^2} - Q(r') + \mathcal{O}(l^2) \right) \quad (46)$$

Therefore:

$$\langle H \rangle_{H,\psi} = \frac{N - 2N_{\text{top}}}{4\pi l^2} \tilde{V}(k=0) + \frac{1}{2} \int d^2r \int d^2r' V(r-r') Q(r) Q(r') \quad (47)$$

We may object that the last term is not the only contribution of order $\mathcal{O}(l^0)$ to $\langle H \rangle_{H,\psi}$. We should in principle include $\mathcal{O}(l^2)$ terms in the local particle density. However, because the density in a filled Landau level is spatially uniform, such terms will appear in $\langle H \rangle_{H,\psi}$ only through their integral over the whole plane. As we have discussed in the end of section 2.2, these integrals are expected to vanish because the total number of electrons $N_e = N - N_{\text{top}}$ contains only the first $\mathcal{O}(l^0)$ correction due to the topological charge of the texture.

Let us now consider the Fock term. Because the Gaussian kernel in eq. (40) is sharply peaked around the origin, with a characteristic length equal to l , it is natural to expand $\text{Tr}(P_\psi(s(r, r')) P_\psi(s(r', r)))$ around $r = r'$. The following integrals are useful:

$$\begin{aligned} \int dx \int dy \frac{1}{\sqrt{x^2 + y^2}} \exp\left(-\frac{x^2 + y^2}{2l^2}\right) &= \pi \sqrt{2\pi} l \\ \int dx \int dy \frac{x^2}{\sqrt{x^2 + y^2}} \exp\left(-\frac{x^2 + y^2}{2l^2}\right) &= \pi \sqrt{\frac{\pi}{2}} l^3 \end{aligned}$$

Keeping the first two terms in the semi-classical expansion of $\langle H \rangle_{F,\psi}$ gives:

$$\langle H \rangle_{F,\psi} = -\frac{e^2}{16\pi\sqrt{2\pi}\epsilon l^3} \int d^2r \left(1 + 2l^2 \text{Tr}(P_0 P_1) + \frac{l^2}{4} \text{Tr}(\nabla P_0 \cdot \nabla P_0)\right) \quad (48)$$

Finally, using eq. (44), the Fock contribution for Coulomb interaction reads:

$$\langle H \rangle_{F,\psi} = -\frac{e^2}{8\sqrt{2\pi}\epsilon l} (N - N_{\text{top}}) + \frac{e^2}{32\pi\sqrt{2\pi}\epsilon l} \int d^2r (\langle \nabla \psi | \nabla \psi \rangle - \langle \nabla \psi | \psi \rangle \langle \psi | \nabla \psi \rangle) \quad (49)$$

Note that the first term has the expected form for the Fock contribution in a fully polarized system, which is equivalent to a system of spin-less fermions. It is negative and proportional to the particle number, in agreement with the physical interpretation that it removes the Coulomb self-interaction of all particles present in the system. The value of the stiffness in the second term is consistent with the value given in the literature [4, 15], in which the Coulomb interaction potential is often written as $V(r) = e^2/\epsilon r$ (Gaussian units).

2.4 Choice of an effective model

So far, we have computed properties of a single Slater determinant $|\mathcal{S}_\psi\rangle$ associated to a smooth spinor ψ . Here, we would like to construct an effective model by considering the family $\mathcal{M}$ of such Slater determinants as an (overcomplete) basis of low energy states. An essential ingredient is the overlap $\langle \mathcal{S}_\psi | \mathcal{S}_{\psi'} \rangle$ between two states in $\mathcal{M}$. The reader is invited to check that the modulus of this overlap can be simply expressed in terms of the corresponding projectors $\hat{P}_\psi$ and $\hat{P}_{\psi'}$:

$$|\langle \mathcal{S}_\psi | \mathcal{S}_{\psi'} \rangle|^2 = \text{Det}(I - \hat{P}_\psi \hat{P}_{\psi'} \hat{P}_\psi) \quad (50)$$

This implies that:

$$\log |\langle \mathcal{S}_\psi | \mathcal{S}_{\psi'} \rangle|^2 = \int \frac{d^2 r}{2\pi l^2} \text{Tr}_{\mathbb{C}^d} \left(\text{Cov. Symb.} \log(I - \hat{P}_\psi \hat{P}_{\psi'} \hat{P}_\psi) \right) \quad (51)$$

The leading order in l^2 is easily extracted by taking the principal symbols, that is by making the approximation $P_\psi \simeq |\psi(r)\rangle\langle\psi(r)|$. At this leading order, the star product becomes the ordinary product, and we have:

$$\log |\langle \mathcal{S}_\psi | \mathcal{S}_{\psi'} \rangle|^2 = \int \frac{d^2 r}{2\pi l^2} \text{Tr}_{\mathbb{C}^d} \log (I - (1 - |\langle\psi(r)|\psi'(r)\rangle|^2) |\psi(r)\rangle\langle\psi(r)|) + \mathcal{O}(l^2) \quad (52)$$

Finally,

$$\log |\langle \mathcal{S}_\psi | \mathcal{S}_{\psi'} \rangle|^2 = \int \frac{d^2 r}{2\pi l^2} \log |\langle\psi(r)|\psi'(r)\rangle|^2 + \mathcal{O}(l^2) \quad (53)$$

This form of the overlap is compatible with a lattice model where each of its N sites hosts a quantum degree of freedom with d independent internal states. Furthermore, Slater determinants associated to textures can be seen as coherent states. To substantiate this claim, let us consider the Hilbert space which is a tensor product of N copies of $\mathbb{C}^d$. Coherent states can be defined as factorizable states of the form $|\mathcal{C}_\psi\rangle = |\psi(1)\rangle \otimes |\psi(2)\rangle \otimes \cdots \otimes |\psi(N)\rangle$, where $|\psi(i)\rangle \in \mathbb{C}^d$ for all i . Note that we have here a redundancy because changing $|\psi(j)\rangle$ into $\exp(i\theta(j))|\psi(j)\rangle$ modifies only the global phase of $|\mathcal{C}_\psi\rangle$. There are least two ways to deal with this. The first one is to lift this ambiguity by fixing a gauge. For example, one may impose that $\psi_1(j)$ be real and positive. The problem with this prescription is that it is useless when $\psi_1(j)$ is equal to zero. To cover the whole projective space $\mathbb{C}P(d-1)$, we have to consider for each site j at least d open subsets characterized by $\psi_i(j) \neq 0$ for $1 \leq i \leq d$ and to patch them together. This procedure is mathematically clean, although it may not be the most convenient for practical calculations. The second way, which we will follow, is to work with unconstrained spinors, but to keep in mind that all physical properties are invariant under local gauge transformations.

The overlap between two such states is given by:

$$\langle \mathcal{C}_\psi | \mathcal{C}_{\psi'} \rangle = \prod_{i=1}^N \langle \psi(i) | \psi'(i) \rangle \quad (54)$$

Taking the squared modulus and transforming slightly, we get:

$$\log |\langle \mathcal{C}_\psi | \mathcal{C}_{\psi'} \rangle|^2 = \sum_{i=1}^N \log |\langle \psi(i) | \psi'(i) \rangle|^2 \quad (55)$$

In the semi-classical limit of slowly varying textures, the discrete texture $|\psi(i)\rangle$ becomes a smooth one $|\psi(r)\rangle$ and we can approximate the sum by an integral, which gives exactly the same leading term as in eq. (53) previously derived.

This discussion motivates the following semi-classical model to describe a system containing $N_{\text{eff}} = N - N_{\text{top}}$ electrons in the lowest Landau level. The semi-classical limit is reached when the typical distance between nearby Skyrmions is large compared to the magnetic length, that is when $N_{\text{top}} \ll N$. The expectation value of the Hamiltonian on

Slater determinants detailed in subsection 2.3 suggests the following form of the energy in this effective model: $\langle H \rangle_\psi \equiv \langle \mathcal{C}_\psi | H | \mathcal{C}_\psi \rangle = \langle H \rangle_{\text{ex},\psi} + \langle H \rangle_{\text{el},\psi}$ where:

$$\langle H \rangle_{\text{ex},\psi} = E_{\text{ex}} \int d^{(2)}r \left(\frac{\langle \nabla \psi | \nabla \psi \rangle}{\langle \psi | \psi \rangle} - \frac{\langle \nabla \psi | \psi \rangle \langle \psi | \nabla \psi \rangle}{\langle \psi | \psi \rangle^2} \right) \quad (56)$$

expresses the short range part of Coulomb interactions (exchange energy) and

$$\langle H \rangle_{\text{el},\psi} = \frac{1}{2} \sum_{ab} \int d^2r \int d^2r' V(r - r') Q(r) Q(r') \quad (57)$$

is the residual long range part due to the spatial variations of the topological charge density $Q(r)$. Eq. (56) has been written for the most general (not necessarily normalized) spinor field $|\psi(r)\rangle$. The presence of the second term in the integrand ensures the expected local gauge invariance of $\langle H \rangle_{\text{ex},\psi}$.

How should we use $\langle H \rangle_\psi$? The first viewpoint is that the family $\mathcal{M}$ of Slater determinants $|\mathcal{S}_\psi\rangle$ associated to smooth spinor fields provides a good starting point for a variational calculation of the ground state for a system of $N - N_{\text{top}}$ electrons (with $N_{\text{top}} \ll N$). This is a very useful approach indeed, which is strongly supported by the fact that it gives the exact ground states for a model with point-like repulsive interactions when $0 \leq N_{\text{top}} \ll N$. We shall dedicate a substantial part of this lecture to such studies, with emphasis on spatially periodic textures (sections 3.3 and 3.4).

But the use of $\langle H \rangle_\psi$ extends beyond the variational determination of ground-states and their properties. Viewing $\mathcal{M}$ as a family of coherent states embedded in the electronic Fock space, $\langle H \rangle_\psi$ can be regarded as the covariant symbol of an effective Hamiltonian operator from which the low energy dynamics can, in principle, be reconstructed. This second viewpoint is in tune with a long tradition in many-body physics. The general idea is that most variational approximations involve a particular continuous family of trial states (such as Slater determinants for Hartree-Fock or general BCS states for Hartree-Fock-Bogoliubov approximations) which can often be regarded as a classical phase-space. The underlying quantum dynamics in electronic Fock space can then be conveniently analyzed from the perspective of coherent state quantization. For several illustrations of this viewpoint, the reader is invited to consult [34, 35, 39]. We shall refer only briefly to coherent state quantization of spin textures, in a discussion of the quantum zero point correction to the variational energy $\langle H \rangle_\psi$ (section 3.2).

A third way to use $\langle H \rangle_\psi$ is somehow intermediate between the two previous ones. In the spirit of time dependent Hartree-Fock approximation, it regards $\langle H \rangle_\psi$ as a classical Hamiltonian. The associated dynamics on $\mathcal{M}$ is expected to approximate rather well the full quantum dynamics. The situation is reminiscent of quantum antiferromagnets. There, strictly speaking, the semi-classical limit is reached when the spin S is large. But it is well known that, at least for magnetically ordered systems, observations on real systems with a small value of S and calculations on a semi-classical expansion in $1/S$ can show a surprisingly good agreement. Most likely, classicality is an emerging property in such systems. Even starting from "extremely quantum" spins $1/2$ at the microscopic level, coarse graining leads to effective spins which tend to behave more and more classically as the spatial scale grows. This analogy with quantum magnets is specially natural for the periodic textures considered in sections 3.3 and 3.4. Their spontaneously broken $SU(d)$ symmetry makes them similar to non-collinear long range ordered antiferromagnets.

Let us now derive the classical equations of motion generated by $\langle H \rangle_\psi$. The first important consideration here is that the time-dependent Schrödinger equation $i\frac{\partial}{\partial t}|\Psi\rangle = H|\Psi\rangle$ can be derived from the following variational principle:

$$\delta \int_{t_i}^{t_f} \left(i\langle \Psi | \frac{\partial \Psi}{\partial t} \rangle - \langle \Psi | H | \Psi \rangle \right) dt = 0 \quad (58)$$

In other words, Schrödinger's equation can be regarded as a classical Hamilton equation in Hilbert space ! The corresponding Hamiltonian function takes the value $\langle \Psi | H | \Psi \rangle$ on state $|\Psi\rangle$. To specify the dynamics, we also need to know the underlying symplectic structure (or Poisson brackets). If M is the dimension of the Hilbert space, the Schrödinger equation reads:

$$\frac{\partial \Psi_a}{\partial t} = -i \frac{\partial \langle \Psi | H | \Psi \rangle}{\partial \bar{\Psi}_a}; \quad \frac{\partial \bar{\Psi}_a}{\partial t} = i \frac{\partial \langle \Psi | H | \Psi \rangle}{\partial \Psi_a}, \quad 1 \leq a \leq M \quad (59)$$

Writing $\Psi_a = (q_a + ip_a)/\sqrt{2}$, with q_a and p_a real variables, we get exactly the usual form of Hamilton's equations:

$$\frac{\partial q_a}{\partial t} = \frac{\partial \langle \Psi | H | \Psi \rangle}{\partial p_a}; \quad \frac{\partial p_a}{\partial t} = -\frac{\partial \langle \Psi | H | \Psi \rangle}{\partial q_a}. \quad (60)$$

A great advantage of variational principles is their ability to deal with constraints. The previous discussion suggests that a reasonable approximation would be to constrain the time evolution to take place within the manifold $\mathcal{M}$ of coherent states $|\mathcal{S}_\psi\rangle$. The variational principle becomes then:

$$\delta \int_{t_i}^{t_f} \left(i \sum_{j=1}^N \langle \psi(j) | \frac{\partial \psi(j)}{\partial t} \rangle - \langle H \rangle_\psi \right) dt = 0 \quad (61)$$

To simplify the writing, we have assumed here that the local spinors are normalized, that is $\langle \psi(j) | \psi(j) \rangle = 1$. Taking the continuum limit $N \rightarrow \infty$, this becomes:

$$\delta \int_{t_i}^{t_f} \left(i \int \frac{d^2 r}{2\pi l^2} \langle \psi(r) | \frac{\partial \psi(r)}{\partial t} \rangle - \langle H \rangle_\psi \right) dt = 0 \quad (62)$$

We shall consider further this Hamiltonian dynamics on $\mathcal{M}$ in sections 3.1 and 4.1 below. A slight modification of it will be useful, in order to deal with the local gauge symmetry sending $|\psi(r)\rangle$ into $f(r,t)|\psi(r)\rangle$ with $f(r,t)$ an arbitrary function. Allowing functions such that $|f|^2 \neq 1$ can be important if we deal with spinors which are not normalized by construction (such as spinors with holomorphic components). Again, this symmetry expresses the invariance of physical quantities with respect to changes in the global factor in front of the wave-function. The original variational principle (58) can be modified slightly to make this generalized gauge symmetry manifest, and this gives:

$$\delta \int_{t_i}^{t_f} \left(i \frac{\langle \Psi | \frac{\partial \Psi}{\partial t} \rangle}{\langle \Psi | \Psi \rangle} - \frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi | \Psi \rangle} \right) dt = 0 \quad (63)$$

The corresponding evolution equation is:

$$i \left(\left| \frac{\partial \Psi}{\partial t} \right\rangle - \frac{\langle \Psi | \frac{\partial \Psi}{\partial t} \rangle}{\langle \Psi | \Psi \rangle} |\Psi\rangle \right) = H|\Psi\rangle - \frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi | \Psi \rangle} |\Psi\rangle \quad (64)$$

This is equivalent to $i|\frac{\partial\Psi}{\partial t}\rangle = H|\Psi\rangle + f(t)|\Psi\rangle$ where $f(t)$ is an arbitrary function. This modified equation has the same physical content as the usual one. To see this, let us diagonalize H in an orthonormal basis $|\alpha\rangle$ so $H|\alpha\rangle = \omega_\alpha|\alpha\rangle$, $1 \leq \alpha \leq M$. If $|\Psi(t)\rangle = \sum_\alpha^M \Psi_\alpha(t)|\alpha\rangle$, we have:

$$\Psi_\alpha(t) = g(t)e^{-i\omega_\alpha t}\Psi_\alpha(0), \quad g(t) = \exp\left(-i\int_0^t f(t')dt'\right) \quad (65)$$

Adapting this to the case of the constrained dynamics on the coherent state manifold $\mathcal{M}$ gives:

$$\delta \int_{t_i}^{t_f} \left(i \int \frac{d^2r}{2\pi l^2} \frac{\langle \psi(r) | \frac{\partial \psi(r)}{\partial t} \rangle}{\langle \psi(r) | \psi(r) \rangle} - \langle H \rangle_\psi \right) dt = 0 \quad (66)$$

We shall use this variational principle in section 4.1 below. It is very close in spirit to the time-dependent Hartree-Fock approximation, with the difference that the wave-function of the system is not allowed to be the most general Slater determinant, but a one of the form $|\mathcal{S}_\psi\rangle$.

2.5 Classical ground-states of the $\mathbb{CP}(d-1)$ model

The previous discussion shows that it is very useful to look for the spinor fields which minimize the variational energy $\langle H \rangle_\psi$. Since the energy scale E_{ex} is of order $e^2/\epsilon l$, we see that $\langle H \rangle_{\text{el},\psi}/\langle H \rangle_{\text{ex},\psi}$ is proportional to $ln^{1/2}$, where n is the average topological charge density. In our semi-classical limit, $ln^{1/2} \ll 1$ and the term $\langle H \rangle_{\text{el},\psi}$ can be treated as a perturbation, with respect to the leading term $\langle H \rangle_{\text{ex},\psi}$. It is therefore natural to concentrate on the $\mathbb{CP}(d-1)$ model defined by the energy functional $\langle H \rangle_{\text{ex},\psi}$. This model has been studied in great detail by field theorists, starting from the seventies. A very pedagogical presentation of many of its properties can be found in Rajaraman's book [40]. Our first concern is to find the minimas of $\langle H \rangle_{\text{ex},\psi}$ with the constraint of a fixed topological charge. To achieve this, we use the Bogomol'nyi-Prasad-Sommerfield inequality which states that:

$$\langle H \rangle_{\text{ex},\psi}/E_{\text{ex}} \geq 2\pi|N_{\text{top}}| \quad (67)$$

To prove this bound in the present case, we start from the following expressions:

$$\frac{\langle H \rangle_{\text{ex},\psi}}{2E_{\text{ex}}} = I + J \quad (68)$$

$$\pi N_{\text{top}} = I - J \quad (69)$$

with:

$$I = \int d^{(2)}r \left(\frac{\langle \partial_{\bar{z}}\psi | \partial_z\psi \rangle}{\langle \psi | \psi \rangle} - \frac{\langle \partial_{\bar{z}}\psi | \psi \rangle \langle \psi | \partial_z\psi \rangle}{\langle \psi | \psi \rangle^2} \right) \quad (70)$$

$$J = \int d^{(2)}r \left(\frac{\langle \partial_z\psi | \partial_{\bar{z}}\psi \rangle}{\langle \psi | \psi \rangle} - \frac{\langle \partial_z\psi | \psi \rangle \langle \psi | \partial_{\bar{z}}\psi \rangle}{\langle \psi | \psi \rangle^2} \right) \quad (71)$$

Since both I and J are positive, we have $I + J \geq |I - J|$, therefore proving the inequality (67). This proof also shows that this lower bound is reached if and only if $J = 0$ for $N_{\text{top}} \geq 0$ or $I = 0$ for $N_{\text{top}} \leq 0$. A sufficient condition for this to hold is $|\partial_{\bar{z}}\psi\rangle = 0$

for $N_{\text{top}} \geq 0$ or $|\partial_z \psi\rangle = 0$ for $N_{\text{top}} \leq 0$. So the variational exchange energy is minimal for analytic textures when holes are added ($N_{\text{top}} \geq 0$) or for anti-analytic ones when particles are added ($N_{\text{top}} \leq 0$). This is striking, because these spaces of analytic (or anti-analytic) textures are quite large. If we fold the plane onto a finite torus, using periodic boundary conditions, they form a dN_{top} -dimensional complex vector space.

It is interesting to apply inequality (67) to the case of point-like interactions. Keeping the first two terms in the l^2 expansion of $\langle H \rangle_\psi$ given in eq. (45), it gives:

$$\langle H \rangle_\psi \geq \frac{W}{4\pi l^2} (|N_{\text{top}}| - N_{\text{top}}) \quad (72)$$

This becomes an equality for analytic spinors when $N_{\text{top}} \geq 0$, so that $\langle H \rangle_\psi = 0$. This is consistent with the form of the Slater determinant $|\mathcal{S}_\psi\rangle$ in this case. Its wave-function, in first quantization, reads:

$$\Psi(r_1 a_1, \dots, r_{N_e} a_{N_e}) = \prod_{i < j} (z_i - z_j) \prod_{i=1}^{N_e} \psi_{a_i}(z_i) e^{-\frac{|z_i|^2}{4l^2}} \quad (73)$$

The first factor prevents two particles from occupying the same position, so this wave-function is an eigenstate of the point-like interaction Hamiltonian with eigenvalue zero [28, 29, 30].

3 Periodic textures

3.1 Perturbation theory for degenerate Hamiltonians

We have just seen that in the manifold $\mathcal{M}$, the ground-states of $\langle H \rangle_{\text{ex}}$ form a rather large submanifold $\mathcal{D}$ composed of analytic textures for $N_{\text{top}} \geq 0$ (and anti-analytic ones when $N_{\text{top}} \leq 0$). What is the effect of the residual interaction $\langle H \rangle_{\text{el}}$ on such system? We know that some care has to be taken while perturbing degenerate systems. The first task is to recast the unperturbed $\langle H \rangle_{\text{ex}} \equiv H_0$ in a form that is convenient for a perturbative analysis. This raises immediate difficulties for degenerate systems, because we have to understand how the ground-state manifold $\mathcal{D}$ behaves with respect to the symplectic structure of $\mathcal{M}$.

Let us pick a point on $\mathcal{D}$, and let us perform a linear analysis of the equations of motion around it. If H_0 is positive and vanishes on $\mathcal{D}$, the Williamson theorem [41] states that we can find canonical coordinates in a neighborhood of this point (taken as the origin) such that the second order Taylor expansion of H_0 at the origin reads:

$$H_0 = \frac{1}{2} \sum_{j=N_0+1}^{N_0+N_d} p_j^2 + \frac{1}{2} \sum_{j=N_0+N_d+1}^N \omega_j (p_j^2 + q_j^2) \quad (74)$$

with $\omega_j > 0$. Here N_0 , N_d , and N_m are non-negative integers such that $N = N_0 + N_d + N_m$ is the total number of degrees of freedom, and hence the dimension of $\mathcal{M}$ is $2N$. In our system of textures, the number of degrees of freedom is dN , where N denoted the number of flux quantas through the system. The conflict of notation should not be a problem, because the discussion of this subsection is mostly conceptual.

Near the origin, $\mathcal{D}$ is defined by the equations:

$$p_{N_0+1} = \dots = p_N = 0 = q_{N_0+N_d+1} = \dots = q_N \quad (75)$$

This leaves $q_1, \dots, q_{N_0+N_d}, p_1, \dots, p_{N_0}$ as independent coordinates on $\mathcal{D}$ near the origin, so the dimension of $\mathcal{D}$ is $2N_0 + N_d$. N_0 , N_d , and N_m will be referred to as the numbers of zero modes, drift motions, and massive modes, respectively. The important remark here is that the dimension of $\mathcal{D}$ doesn't fix separately the values of N_0 and N_d . These values can be extracted by a different procedure [42] which we outline here.

Phase-space geometry is not Euclidean (because there is no distance invariant under canonical transformations) but symplectic. Its basic object is not a metric, but a rank two antisymmetric form $\omega = \sum_{j=1}^N dp_j \wedge dq_j$. This form assigns to two infinitesimal vectors $(\delta p_i, \delta q_i), (\delta' p_i, \delta' q_i)$ the number $\omega(\delta, \delta') = \sum_{j=1}^N (\delta p_j \delta' q_j - \delta q_j \delta' p_j)$. This form is invariant under canonical transformations and in particular, under Hamiltonian evolutions. If $N = 1$, this conservation law is just the Liouville theorem on the conservation of phase-space volume. The invariance of ω shows that the notion of orthogonality of two infinitesimal vectors (in the sense of ω) has an intrinsic meaning. A simple inspection shows that the tangent vectors at the origin which are orthogonal to all tangent vectors along $\mathcal{D}$ correspond to $\delta p_1 = \dots = \delta p_{N_0+N_d} = 0 = \delta q_1 = \dots = \delta q_{N_0}$. This implies that the restriction of ω to the tangent space of $\mathcal{D}$ at the origin contains a d -dimensional subspace of vectors which are orthogonal to all tangent vectors along $\mathcal{D}$. This subspace (the kernel of the restriction of ω) is spanned by the drift motions, for which the only non-zero components of the velocity are $\dot{q}_{N_0+1}, \dots, \dot{q}_{N_0+N_d}$. Such motions are generated by Hamiltonians of the form $\sum_{j=N_0+1}^{N_0+N_d} a_j p_j$. Their physical importance is that, unlike small oscillations associated to massive modes, they are in principle unbounded, which makes them qualitatively different from both zero modes and massive modes.

This local analysis is interesting, but it raises immediately the question whether it can be extended to a larger region (i.e. an open subset in $\mathcal{M}$) intersecting $\mathcal{D}$. The answer is positive, with the assumption that N_d should be constant along the intersection of $\mathcal{D}$ with this region. If this condition is satisfied, we can find canonical coordinates such that a normal form similar to (74) holds, with two differences. First, the massive mode frequencies ω_j may vary along $\mathcal{D}$ so that ω_j becomes a function of $q_1, \dots, q_{N_0+N_d}, p_1, \dots, p_{N_0}$. Second, the quadratic kinetic term associated to drift motion is not necessarily diagonal, and is replaced by $\frac{1}{2} \sum_{i=N_0+1}^{N_0+N_d} \sum_{j=N_0+1}^{N_0+N_d} (g^{-1})_{ij} p_i p_j$, where $g_{ij}(q_1, \dots, q_{N_0+N_d}, p_1, \dots, p_{N_0})$ is a metric tensor on $\mathcal{D}$.

I am not going to discuss the proof of this result here, because I fear that it may be of limited interest to most readers. We expect that this should be a direct consequence of the relative Darboux theorem for a submanifold $\mathcal{D}$ of a symplectic manifold $\mathcal{M}$ [44]. One version of this theorem states that if coordinates (p'_i, q'_i) exist such that $\mathcal{D}$ is defined by the equations $p'_{N_0+1} = \dots = p'_N = 0 = q'_{N_0+N_d+1} = \dots = q'_N$, and that ω takes its canonical form on $\mathcal{D}$, there exists a smooth one to one transformation from (p'_i, q'_i) to *canonical coordinates* (p_i, q_i) on $\mathcal{M}$ which acts like the identity on $\mathcal{D}$. Given this relative Darboux theorem, the main task seems to be to prove the existence of the coordinates (p'_i, q'_i) with the desired properties. This is easy in two particular cases. When $N_d = 0$, the restriction of ω to $\mathcal{D}$ is non degenerate, so the existence of these coordinates is provided by the usual Darboux theorem applied to the submanifold $\mathcal{D}$. When $N_0 = 0$, all pairs of tangent vectors along $\mathcal{D}$ at an arbitrary point of $\mathcal{D}$ are orthogonal, so the restriction of ω to $\mathcal{D}$ is zero. The existence of the required coordinates follows, because we impose only the value of ω on $\mathcal{D}$. For intermediate values of N_d (between 1 and $\dim \mathcal{D} - 1$), we have to prove that the distribution of N_d -dimensional subspaces on $\mathcal{D}$ obtained by taking the

kernel of the restriction of ω is integrable, i.e. that this kernel coincides with the tangent space to the submanifolds defined by fixing the values of all coordinates, excepted $q'_{N_0+1}, \dots, q'_{N_0+N_d}$. I believe that this is true [45], and probably well known to mathematicians, although I haven't been able to locate a proof in the literature.

Coming back to our problem, we have a rather good numerical evidence (but no conceptual proof, unfortunately) that in the case where $\mathcal{D}$ is the submanifold of analytic textures, we have $N_d = 0$. In this situation, we may therefore write:

$$H_0 = \frac{1}{2} \sum_{j=N_0+1}^N \omega_j(p_j^2 + q_j^2) \equiv \sum_{j=1}^{N_m} \omega_j(p_s, q_s) J_j \quad (76)$$

Here, we have introduced the slow variables $(p_s, q_s) \equiv (p_1, \dots, p_{N_0}, q_1, \dots, q_{N_0})$ which are good canonical coordinates on $\mathcal{D}$, and action variables J_j for massive modes.

For this Hamiltonian, the J_j 's associated to massive modes are integrals of motion. When they are non zero, they can induce a motion along $\mathcal{D}$, because Hamilton's equation read for $1 \leq j \leq N_0$:

$$\dot{q}_j = \sum_{k=1}^{N_m} \frac{\partial \omega_k}{\partial p_j}(p_s, q_s) J_k \quad (77)$$

$$\dot{p}_j = - \sum_{k=1}^{N_m} \frac{\partial \omega_k}{\partial q_j}(p_s, q_s) J_k \quad (78)$$

At the classical level, we may take $J_k = 0$ for all k and no motion along $\mathcal{D}$ is generated. Quantum mechanically, this is no longer true, and we can no longer ignore the quantum zero point energy correction H_{qzpc} coming from the massive modes. Its effect is to lift the degeneracy between the classical states lying on $\mathcal{D}$, and to favor the minimas of H_{qzpc} . A more detailed discussion of this quantum correction is given in Appendix 5.4.

Let us now switch on a small perturbation H_1 , which it is natural to expand in powers of the massive mode action-angle coordinates $p_{N_0+j} = \sqrt{2J_j} \cos \theta_j$, $q_{N_0+j} = \sqrt{2J_j} \sin \theta_j$:

$$H_1 = H_1^{(0)} + \sum_{j=1}^{N_m} J_j^{1/2} (a_j(p_s, q_s) e^{i\theta_j} + \bar{a}_j(p_s, q_s) e^{-i\theta_j}) + \mathcal{O}(J_j) \quad (79)$$

where $H_1^{(0)}$ and a_j are of order ϵ .

At first order in ϵ , we may keep only the term $H_1^{(0)}$ in H_1 . Indeed, the term proportional to $J_j^{1/2}$ corresponds to a constant driving force which pulls the massive coordinates away from their unperturbed equilibrium value at $J_j = 0$. This shift induces a change in the energy of the massive mode j of order $|a_j|^2/\omega_j$ which is proportional to ϵ^2/ω_j and therefore higher order in ϵ than the $H_1^{(0)}$ term. We see also that this induced ϵ^2 contribution to the effective Hamiltonian is smaller when the characteristic frequencies associated to massive modes are large. To summarize, the first corrections due to the H_1 perturbation are captured by the effective Hamiltonian:

$$H_{\text{eff}} = H_{\text{qzpc}}(p_s, q_s) + H_1^{(0)}(p_s, q_s) \quad (80)$$

We emphasize once again that it acts on the degenerate manifold $\mathcal{D}$, which is symplectic because $N_d = 0$, so it constitutes a good classical phase-space. On this manifold, H_{eff} is

the sum of the quantum zero point correction due to the massive modes of H_0 and the perturbation H_1 , restricted to $\mathcal{D}$.

These considerations motivate us to discuss briefly the spectrum of massive oscillators in the vicinity of a periodic analytic texture, which therefore belongs to $\mathcal{D}$, and was chosen to minimize the perturbation $H_1^{(0)}(p_s, q_s) \equiv \langle H_{\text{el}} \rangle_\psi$. The construction of this optimal texture will be presented in section 3.3.

3.2 Remarks on the Hessian of the exchange energy

To be specific, we assume a positive topological charge. The discussion of section 2.5 shows that the variational energy can be conveniently be written as:

$$\langle H \rangle_{\text{ex}, \psi} / E_{\text{ex}} = 2\pi N_{\text{top}} + 4J \quad (81)$$

where J is defined by eq. (71). Consider now small deviations $|\psi\rangle \rightarrow |\psi\rangle + \sqrt{\langle \psi | \psi \rangle} |\phi\rangle$ away from analytic spinor $|\psi\rangle$. Inserting this ansatz inside eq. (81) gives directly the second derivative (Hessian operator) of $\langle H \rangle_{\text{ex}, \psi}$:

$$\langle H \rangle_{\text{ex}, \psi} / E_{\text{ex}} = 2\pi |N_{\text{top}}| + 4\langle \phi | M^+ P M | \phi \rangle + \dots \quad (82)$$

Here, we have introduced two operators M and P acting on the spinor fields ϕ describing small deviations. Explicitly:

$$M|\phi\rangle = |\partial_z \phi\rangle + \frac{1}{2} \frac{\langle \partial_z \psi | \psi \rangle}{\langle \psi | \psi \rangle} |\phi\rangle \quad (83)$$

$$P|\phi\rangle = |\phi\rangle - \frac{|\psi(z)\rangle \langle \psi(z)|}{\langle \psi(z) | \psi(z) \rangle} |\phi\rangle \quad (84)$$

A very important property of the M operator is that:

$$[M, M^+] = \frac{1}{2} \mathcal{B}(r) = \pi Q(r) \quad (85)$$

Here $\mathcal{B} = \partial_x \mathcal{A}_y - \partial_y \mathcal{A}_x$ is the gauge-invariant flux density associated to the Berry connection. If $\mathcal{B}(r)$ constant, the spectrum of $M^+ M$ is $\{\frac{\mathcal{B}}{2} n, n = 0, 1, 2, \dots\}$. As we are going to show in the next subsection, the spatial variations of $\mathcal{B}(r)$ are quite small for the optimal periodic texture. The residual inhomogeneities are not expected to close the gaps in this Landau-level like spectrum. What is more difficult to analyze is the effect of the projector P . It imposes the local variation $|\phi(r)\rangle$ to be orthogonal to the reference spinor $|\psi(r)\rangle$. This is certainly a serious perturbation for small values of d . At large d , the probability for two randomly chosen spinors to be orthogonal becomes large, so we may expect that the effect of P is small in the large d limit. Most likely, the Hessian of $\mathbb{C}P(d-1)$ model is gapped, with an energy gap of order $\frac{e^2}{4\pi\epsilon l} n l^2$, where $Q(r) = n$. A good numerical evidence that this is indeed true has been obtained recently by D. Kovrizhin, using the dynamics given by eq. (66). His results for $d = 3$, with the reference spinor described as in sections 3.3 and 3.4 are shown on Fig. 3.

In view of eq. (80), we need to estimate the quantum zero point energy correction due to the massive modes of $\langle H \rangle_{\text{ex}}$. The form of the Hessian, eq. (82) shows that it is bilinear in $\phi_a(r)$, $\bar{\phi}_{a'}(r')$. The discussion in Appendix 5.4 suggests then that H_{qzpc} should

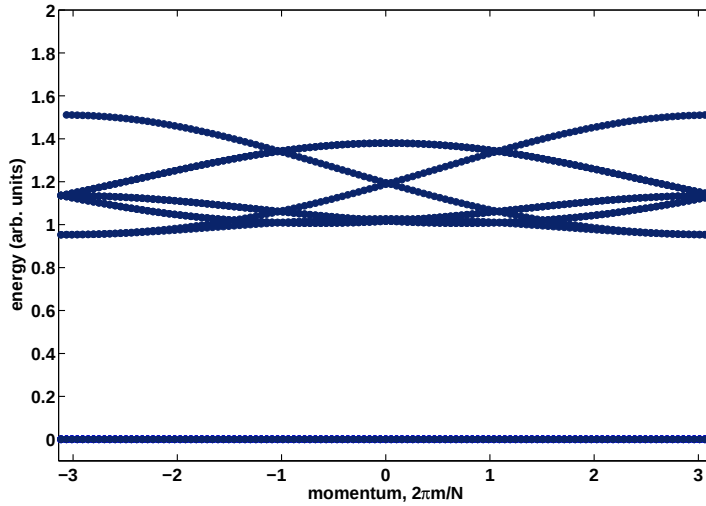


Figure 3: The spectrum of the Hessian in the $d = 3$ case. The flat branch at zero frequency corresponds to variations within the analytic subspace. The first Landau-like level is broadened by the combined effect of the periodic modulation of the Berry flux $\mathcal{B}(r)$ and of the projector P . Nevertheless, there is a clear gap between the degenerate zero energy level and this broadened first Landau level.

vanish, because it is only sensitive to squeezing operators, which would be detected by the presence of quadratic terms of the form $\phi_a(r)\phi_{a'}(r')$ or $\bar{\phi}_a(r)\bar{\phi}_{a'}(r')$ in the Taylor expansion of the covariant symbol $\langle H \rangle_{\text{ex}}$. But such terms are clearly absent from eq. (82). At this stage, we therefore conjecture that $H_{\text{qzpc}} = 0$. A sound mathematical theory of the family $|\mathcal{S}_\psi\rangle$, viewed as coherent states in the fermionic Fock space doesn't seem to be available, in part because of the infinite dimensionality of the space of possible smooth textures. Physically, we expect the presence of a spatial cut-off below the magnetic length, so we might be able to get back to the more familiar situation of a finite-dimensional family. Besides the previous formal argument, our conjecture seems to be supported by two different observations. It has indeed been found by a combination of numerical and analytical studies that quantum corrections to the effective energy functional introduced in section 2.4 are small [46]. Second, as observed long ago, analytic textures are exact zero energy eigenstates for a model with point-like repulsive interaction [28, 29, 30], and therefore their degeneracy is preserved to all orders in quantum fluctuations. Of course, it is not clear whether this conclusion, valid for the model with point-like repulsion, can be transferred to the quantum $\mathbb{C}P(d-1)$ model. In section 2.3, we have computed to $\mathcal{O}(l^2)$ correction to the expectation value of the interaction energy, and this led to the $\mathbb{C}P(d-1)$ energy functional, eq. (45). I don't know if the next corrections arising from both Hartree and Fock terms will cancel. Unfortunately, the computation of the Fock term requires the knowledge of P_2 in the series expansion of P_ψ , whose expression is a priori quite complicated. So the precise relationship between the quantum $\mathbb{C}P(d-1)$ model and the model with point-like repulsion remains an interesting open question.

3.3 Variational procedure for energy minimization

Let us consider the case $0 < N_{\text{top}} \ll N$, so that $\langle H \rangle_{\text{ex}, \psi}$ is minimized for analytic textures. Neglecting the quantum zero point correction coming from the finite frequency modes of the Hessian of $\langle H \rangle_{\text{ex}}$, we have then to minimize $\langle H \rangle_{\text{el}, \psi}$ with the constraint of a fixed topological charge. Intuitively, the Coulomb interaction being repulsive, one would like to make the topological charge density as uniform as possible. This motivates a variational search within the class of periodic textures. Let us then pick two independent vectors γ_1 and γ_2 on the plane. Our goal is to construct holomorphic spinor fields $\psi_a(z)$ such that all physical properties of the corresponding texture are periodic under translations by γ_1 and γ_2 . This seems first impossible, because the only holomorphic functions which have such double periodicity are constants. However, we should view $|\psi(z)\rangle$ as a representative of the complex line it generates, so the appropriate notion of periodicity is that for any lattice vector $\gamma = n_1\gamma_1 + n_2\gamma_2$ (n_1, n_2 integers) there should be a holomorphic function $f_\gamma(z)$ such that:

$$|\psi(z + \gamma)\rangle = f_\gamma(z)|\psi(z)\rangle \quad (86)$$

A particularly important class of functions satisfying this is the family of θ functions, which satisfy $f_\gamma(z) = \exp(a_\gamma z + b_\gamma)$, a_γ and b_γ being complex numbers, which are functions of the lattice vector γ . These two functions are called the type of the θ function. A remarkable mathematical result states that any periodic holomorphic map from the complex plane to $\mathbb{CP}(d-1)$ is obtained, up to a gauge transformation, from a holomorphic spinor field whose d components are θ functions of the same type [47]. An important example of θ functions is:

$$\theta_p(z) = \sum_n e^{i(\pi\tau d(n-p/d)(n-1-p/d) + 2\sqrt{d}(n-p/d)z)} \quad (87)$$

Here, we have chosen $\gamma_1 = \pi\sqrt{d}$, $\gamma_2 = \pi\sqrt{d}\tau$, with $\Im\tau > 0$, and p is an integer. These functions satisfy:

$$\theta_p(z + \gamma_1) = \theta_p(z) \quad (88)$$

$$\theta_p(z + \gamma_2) = e^{-i2\sqrt{d}z} \theta_p(z) \quad (89)$$

In particular, we have $a_{\gamma_1} = 0$ and $a_{\gamma_2} = -i2\sqrt{d}$. Let us compute the topological charge of such texture enclosed in the parallelogram $\mathcal{P}(\gamma_1, \gamma_2)$ spanned by the two basis vectors. This is done easily because:

$$\left(\frac{\langle \psi | \partial_z \psi \rangle}{\langle \psi | \psi \rangle} \right) (z + \gamma) = a_\gamma + \left(\frac{\langle \psi | \partial_z \psi \rangle}{\langle \psi | \psi \rangle} \right) (z) \quad (90)$$

From this, we deduce that:

$$\frac{1}{2\pi} \oint_{\mathcal{P}(\gamma_1, \gamma_2)} \mathcal{A}.dl = \frac{1}{2\pi i} (a_{\gamma_1} \gamma_2 - a_{\gamma_2} \gamma_1) \quad (91)$$

The topological charge enclosed in $\mathcal{P}(\gamma_1, \gamma_2)$ is then equal to d . A similar calculation shows that the phase of these θ functions winds by $2\pi d$ when one goes counterclockwise around $\mathcal{P}(\gamma_1, \gamma_2)$. This domain contains therefore d zeros for each θ function of the above type. Another remarkable result is the special case of Riemann-Roch theorem for a complex

torus. It states that the θ functions of a given type form a complex vector space, whose dimension is finite and is equal to the topological charge within $\mathcal{P}(\gamma_1, \gamma_2)$. For the type given by eqs. (88), (89), a possible basis is the set of θ_p 's for $0 \leq p \leq d-1$. The positions of zeros in $\mathcal{P}(\gamma_1, \gamma_2)$ are illustrated on Fig. 4.

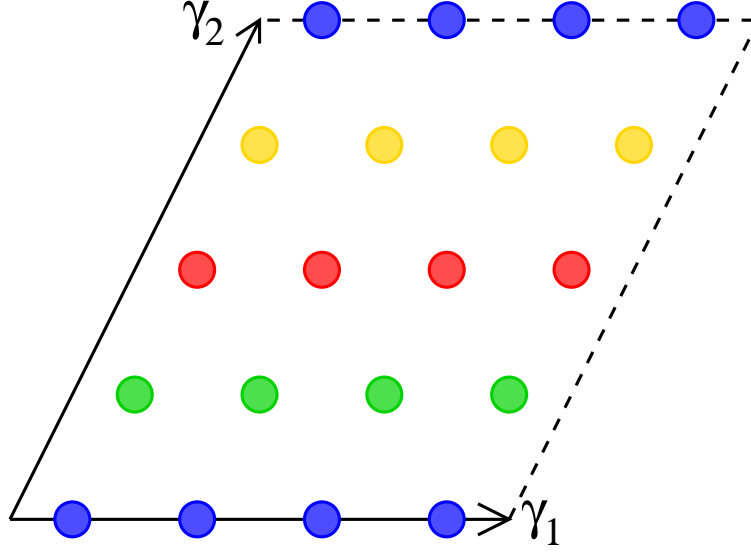


Figure 4: The location of zeros of the basis θ_p functions for $d = 4$. Colors correspond to $p = 0$ (blue), $p = 1$, (green), $p = 2$ (red), and $p = 3$ (yellow).

These properties are very reminiscent of the problem of a quantum particle on a torus in the presence of a uniform magnetic field [48]. This is not a coincidence, because quantum wave functions in the lowest Landau level are analytic functions multiplied by $\exp(-|z|^2/4l^2)$. To fold the plane onto a torus, periodic boundary conditions are imposed by fixing the eigenvalues of the magnetic translation operators along γ_1 and γ_2 , which commute only if $\mathcal{P}(\gamma_1, \gamma_2)$ encloses a finite number of flux quantas Φ/Φ_0 . These boundary conditions impose that the analytic factors in the admissible wave-functions are θ functions of a prescribed type, with the corresponding topological charge equal to Φ/Φ_0 .

It is also useful for our purposes to understand the action of translations on θ functions. We have:

$$\theta(z - w + \gamma) = \exp(a_\gamma z + (b_\gamma - a_\gamma w))\theta(z - w) \quad (92)$$

This shows that the translated function $z \rightarrow \theta(z - w)$ is again a θ function. Whereas a_γ is preserved in this operation, b_γ is in general modified. To preserve also b_γ , we have to multiply the transformed function $\theta(z - w)$ by an exponential factor. We set then the following definition:

$$\mathcal{T}_w \theta(z) = e^{\mu(w)z} \theta(z - w) \quad (93)$$

This new transformation preserves b_γ if and only if:

$$\mu(w)\gamma - a_\gamma w \in 2\pi\mathbb{Z} \quad (94)$$

Because this holds in particular for both γ_1 and γ_2 , the possible values of w are discrete, and they correspond to the points of a lattice generated by γ_1/d and γ_2/d :

$$w(n_1, n_2) = \frac{n_1}{d}\gamma_1 + \frac{n_2}{d}\gamma_2 \quad (95)$$

$$\mu(w) = \frac{n_1}{d}a_{\gamma_1} + \frac{n_2}{d}a_{\gamma_2} \quad (96)$$

As in the case of a particle in the lowest Landau level, this is a projective representation of translations, and more precisely:

$$\mathcal{T}_w \mathcal{T}_{w'} = e^{i\frac{2\pi}{d}(m_1 m'_2 - m_2 m'_1)} \mathcal{T}_{w'} \mathcal{T}_w \quad (97)$$

Note that the phase factor $(m_1 m'_2 - m_2 m'_1)/d$ has a simple meaning: it is the topological charge inside the parallelogram delimited by w and w' . These translations have a simple action on the basis function θ_p :

$$\mathcal{T}_{\frac{\gamma_1}{d}} \theta_p = e^{i\frac{2\pi p}{d}} \theta_p \quad (98)$$

$$\mathcal{T}_{\frac{\gamma_2}{d}} \theta_p = \lambda \theta_{p+1} \quad (99)$$

where $\lambda = \exp(-i\pi\tau(d+1/d))$.

Given all these preparations, we see that the general periodic texture with periods γ_1 and γ_2 takes the form:

$$\psi_a(r) = \sum_{b=0}^{d-1} M_{ab} \theta_b \quad (100)$$

where M_{ab} are the complex entries of a $d \times d$ matrix M . The global $SU(d)$ symmetry manifests itself through the invariance of $\langle H \rangle_{\text{el}}$ under the transformation $M \rightarrow UM$ where U is an arbitrary unitary matrix.

3.4 Properties of periodic textures

After a substantial amount of numerical work, due to Dima Kovrizhin, it appears that the variational energy $\langle H \rangle_{\text{el}}$ is minimal for $\tau = \exp(i\pi/3)$, and $M = I$ (the identity matrix). This corresponds to a spontaneous symmetry breaking of the global $SU(d)$ symmetry. In fact all unitary matrices M are ground-states of $\langle H \rangle_{\text{el}}$. The optimal value of τ corresponds to a triangular lattice of Skyrmions. Such a state has been represented for $d = 2$ on Fig. 2.

An interesting property of these periodic textures is that the spatial modulation of the topological charge density is more periodic than the ansatz $M = I$ suggests at first glance, in the sense that its elementary periods are γ_1/d and γ_2/d . At large d the modulation contains mostly the lowest harmonic, and its amplitude decays exponentially with d . The large d behavior can be computed explicitly for a square lattice:

$$Q(x, y) \simeq \frac{2}{\pi} - 4de^{-\pi d/2} [\cos(2\sqrt{d}x) - 2e^{-\pi d/2} \cos^2(4\sqrt{d}x) + (x \leftrightarrow y)] + \dots \quad (101)$$

A picture of these spatial modulations is shown on Fig. 5.

Only the triangular lattice seems to yield a true local energy minimum. This is most directly seen by computing eigenfrequencies of small deformation modes, using the method to be described below.

4 Collective excitations around periodic textures

4.1 Time dependent Hartree-Fock equations

Let us now turn to collective excitations around such textures. What we are going to do is very reminiscent of the traditional linear spin-wave theory in quantum magnetic

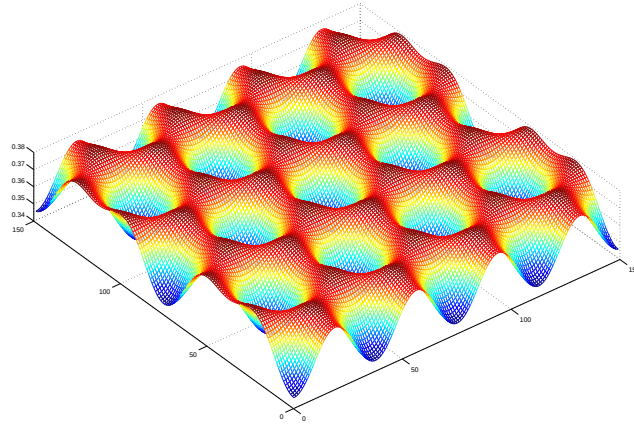


Figure 5: The spatial variations of the topological charge density for the optimal periodic crystal with $d = 4$.

systems. We shall use the variational formulation of the quantum dynamics given by eq. (66), where the variation of $|\psi(r, t)\rangle$ has to be taken within the subspace of analytic spinors. To achieve this, it is convenient to work on a system with finite volume, to keep phase-space dimension finite. One way to do this is to introduce a large supercell $\mathcal{P}(N_1\gamma_1, N_2\gamma_2)$, containing dN_1N_2 topological charges. We fix the type for the θ functions across this supercell. The corresponding allowed translations take the form:

$$w(m_1, m_2) = \frac{m_1}{dN_2}\gamma_1 + \frac{m_2}{dN_1}\gamma_2 \quad (102)$$

$$\mu(m_1, m_2) = \frac{m_1}{dN_2}a_{\gamma_1} + \frac{m_2}{dN_1}a_{\gamma_2} \quad (103)$$

These can be used to construct a basis of dN_1N_2 θ functions:

$$\chi_{p,m_1,m_2} = \mathcal{T}_{w(m_1,m_2)}\theta_p, \quad 1 \leq m_1 \leq N_2 - 1, \quad 1 \leq m_2 \leq N_1 - 1 \quad (104)$$

Now, it is important to note that the type of these θ functions across the elementary cell $\mathcal{P}(\gamma_1, \gamma_2)$ depends on (m_1, m_2) since:

$$\frac{\chi_{p,m_1,m_2}(z + \gamma)}{\chi_{p,m_1,m_2}(z)} = e^{(a_\gamma z + b_\gamma + c_\gamma)} \quad (105)$$

and for example:

$$c_{\gamma_1} = -i2\pi \frac{m_2}{N_1} \quad c_{\gamma_2} = i2\pi \frac{m_1}{N_2} \quad (106)$$

We see that the type associated to $\mathcal{P}(\gamma_1, \gamma_2)$ cell is N_2 -periodic in m_1 and N_1 -periodic in m_2 : somehow we recover a notion of *Brillouin zone* although we do not have periodicity in the usual sense (as for plane waves or Bloch functions), because we are dealing with analytic functions.

We are now looking for small deviations away from the optimal texture, assuming that:

$$\psi_a(z, t) = \theta_a(z, t) + \sum_{p,m_1,m_2} M_{a,p,m_1,m_2}(t)\chi_{p,m_1,m_2}(z) \quad (107)$$

In the spirit of spin-wave theory, we assume here that the amplitudes $M_{a,p,m_1,m_2}(t)$ are infinitesimals of order one. Using the fact that the topological charge density in the reference texture is γ_1 and γ_2 periodic, we deduce that the linearized equations of motions couple (m_1, m_2) only to itself and to $(-m_1, -m_2)$. This is very similar to what we would get in a superfluid or in a quantum antiferromagnet if we could identify (m_1, m_2) with the momentum of the excitation. The same structure has been obtained for the Bogoliubov theory of collective modes in superfluids in the presence of a vortex lattice [49]. The new feature here is the presence of internal degrees of freedom, leading to matrix eigenvalue equations of size $2d^2 \times 2d^2$, but because of the high symmetry of the $Q(r)$ profile, and in particular its γ_1/d and γ_2/d periodicity, this large matrix structure breaks into small blocks of size 2 by 2, which makes the computation of the collective mode spectrum much easier !

4.2 Collective mode spectrum

To analyze the corresponding collective mode spectrum, let us first concentrate on the zero-momentum sector $(m_1, m_2) = (0, 0)$ which yields a Hamiltonian system with $N = d^2$ degrees of freedom. As we have seen, this sector exhibits a d^2 dimensional ground-state manifold obtained by letting the unitary group act on the reference texture, given by $M = I$. To proceed further, it is useful to return to the general analysis of section 3.1 concerning the classical dynamics in the vicinity of a degenerate ground-state manifold $\mathcal{D}$. There, we introduced a normal form characterized by three integers, N_0 , N_d , and N_m . Recall that the dimension of $\mathcal{D}$ is $2N_0 + N_d$. What is the effect of changing the value of N_d while keeping the dimension of $\mathcal{M}$ and $\mathcal{D}$ fixed? Let us show, on few examples that it affects qualitatively the structure of the linearized equations of motion.

The simplest case is $\dim \mathcal{D} = 1$. This is realized for $(N_0, N_d, N_m) = (0, 1, 0)$. An example of this is provided by a free particle in one dimension with the Hamiltonian: $H = \frac{1}{2}P^2$. Here $\mathcal{D}$ is the X axis and the equations of motion read:

$$\begin{pmatrix} \dot{X} \\ \dot{P} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X \\ P \end{pmatrix} \quad (108)$$

An important feature is that this matrix is not diagonalizable, it forms a Jordan block with eigenvalue zero. In physical terms, this means that moving away by ϵ along the P axis generates drift motion parallel to $\mathcal{D}$ with velocity ϵ .

Next, let us consider $\dim \mathcal{D} = 2$. The first possibility is to have $(N_0, N_d, N_m) = (0, 2, 0)$. Let us assume that $\mathcal{D}$ is the X_1, X_2 plane. A simple choice for H is: $H = \frac{1}{2}P_1^2 + \frac{1}{2}P_2^2$, with the corresponding equations of motion:

$$\begin{pmatrix} \dot{X}_1 \\ \dot{P}_1 \\ \dot{X}_2 \\ \dot{P}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ P_1 \\ X_2 \\ P_2 \end{pmatrix} \quad (109)$$

There are now two Jordan blocks, one for each flat direction along $\mathcal{D}$. This situation is characterized by the fact that generating functions of drift motions, P_1 and P_2 commute everywhere, and in particular on the ground-state subspace.

A second possibility with two degrees of freedom is also $(N_0, N_d, N_m) = (1, 0, 1)$. As an example, we may take $\mathcal{D}$ to be the X_1, P_1 plane, and the hamiltonian $H = \frac{\omega}{2}(X_2^2 + P_2^2)$ with the equations of motion:

$$\begin{pmatrix} \dot{X}_1 \\ \dot{P}_1 \\ \dot{X}_2 \\ \dot{P}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ P_1 \\ X_2 \\ P_2 \end{pmatrix} \quad (110)$$

Here, we have only one zero eigenvector for each flat direction and there is no Jordan block. Besides, there is a finite frequency oscillator. The qualitative difference with the previous case is that generating functions of drift motions, X_1 and P_1 do not commute on $\mathcal{D}$.

It is instructive to see how these two possibilities can be realized in a simple system with two classical spins. We choose $H = \vec{S}_1 \cdot \vec{S}_2$, with the constraints $||\vec{S}_1||^2 = s_1$, and $||\vec{S}_2||^2 = s_2$. The ground-state manifold $\mathcal{D}$ is easily obtained. It is parametrized by a unit vector $\vec{n}$ such that $\vec{S}_1 = s_1\vec{n}$, $\vec{S}_2 = -s_2\vec{n}$. So $\mathcal{D}$ is a two-dimensional sphere. Because of the global spin rotation symmetry, we can always induce motions along $\mathcal{D}$ by taking the components of the total angular momentum operator as generators. The classical equations of motion:

$$\frac{d\vec{S}_i}{dt} = (\vec{S}_1 + \vec{S}_2) \wedge \vec{S}_i \quad (111)$$

exhibit the eigenfrequencies $\{0, 0, s_1 - s_2, s_2 - s_1\}$. We see that something special happens when the two spins have the same length. In the general case $s_1 \neq s_2$, so that $\vec{S}_1 + \vec{S}_2 \neq 0$ on $\mathcal{D}$. Because of this, the generators of global rotations do not commute on $\mathcal{D}$, which shows that we are in the second case, according to the previous discussion. This is consistent with the fact that we have a massive mode here.

The case $s_1 = s_2$ is special in that the generators of global rotations commute on $\mathcal{D}$, so we are in the first case with two Jordan blocks. Physically, these drift motions correspond to a precession of both spins along $\vec{S}_1 + \vec{S}_2$, which is small but non zero for small deviations away from $\mathcal{D}$ in the system phase-space. In mathematical terms, in the vicinity of $\mathcal{D}$, the symplectic structure on $\mathcal{M}$ is similar to the familiar one on the cotangent bundle over $\mathcal{D}$. In this situation, physicists prefer usually the Lagrangian formulation, which takes the form here:

$$L \propto (\partial_t \vec{n})^2 \quad (112)$$

This case is very close to what happens in a Néel antiferromagnet, and is illustrated on Fig. 6

Let us now come back to the collective dynamics around the optimal texture. We can show that we are very much like in the later case with rotations replaced by $SU(d)$ transformations. We consider the infinitesimal one corresponding to the anti-hermitian matrix ξ . It is easy to show that the generator of this transformation is the functional:

$$\Phi_\xi = i \int d^2r \sum_{a,b} \frac{\bar{\psi}_a(r) \xi_{ab} \psi_b(r)}{\langle \psi(r) | \psi(r) \rangle} \quad (113)$$

Using the symmetries between the basis θ functions $\theta_p(r)$, such as eqs. (98) and (99), it is easy to show that on the optimal texture for which $\psi_p(r) = \theta_{p-1}(r)$, the integral gives a

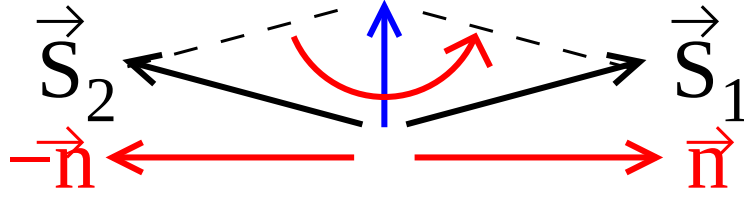


Figure 6: Illustration of drift motion for a spin configuration (black arrows) close to the antiferromagnetic ground state. Spins precess around the total magnetization (blue arrow) at an angular velocity that is proportional to the deviation away from collinearity.

contribution proportional to $\sum_{a,b} \delta_{ab} \xi_{ab} = \text{Tr } \xi = 0$ for ξ in the Lie algebra of $SU(d)$. So we have $N_d = d^2 - 1$ and therefore we expect exactly $d^2 - 1$ Jordan blocks for the linearized equations of motion in the zero momentum sector. This has been confirmed by detailed numerical studies for $d \in \{2, 3, 4\}$. The missing degree of freedom corresponds to diagonal generators which make the difference between the Lie algebras of $U(d)$ and $SU(d)$. But the equations of motion are ill defined in this 2-dimensional block, in agreement with the fact that it corresponds to uniform gauge transformations.

Let us now consider a small but finite momentum (in the sense of (m_1, m_2)), which may be treated as a small perturbation. When it acts on any of the $d^2 - 1$ Jordan blocks obtained for zero momentum, we observe that this block disappears, and gives rise to a pair of opposite eigen-frequencies, which grow linearly with momentum. Putting all the perturbed blocks together, we have $d^2 - 1$ Goldstone branches which disperse linearly. There remains another mode, which originates from the diagonal generators at zero momentum, whose dispersion is not always linear in momentum, and depends on the shape of the long-range interaction. For a repulsive potential $V(r) \propto r^{-\alpha}$, the corresponding dispersion relation is $\omega \propto k^{1+\alpha/2}$. This is reminiscent of the dispersion of magnetophonons in a 2D Wigner crystal [50].

4.3 Towards an effective sigma model description

Can we go beyond this linearized dynamics? The analogy with antiferromagnets suggests that it should be possible. In these systems, it is well known that the key to this upgrade beyond linear spin wave theory is to replace an expansion in *small deviations* away from the reference ordered state by a *gradient expansion*, which is, in a non-linear setting, reminiscent of the perturbation theory of Jordan blocks by a small momentum performed in the linear case. In the case of Néel order, we get the $O(3)$ non-linear sigma model, whose Lagrangian density reads:

$$L \propto (\partial_t \vec{n})^2 - (\partial_x \vec{n})^2 - (\partial_y \vec{n})^2 \quad (114)$$

Clearly this generalizes eq. (112) to slowly varying configurations. In the $SU(d)$ case, we should expect an expression of the form :

$$L \propto \text{Tr} (\partial_t g A \partial_t g^+ - \partial_x g B_x \partial_x g^+ - \partial_y g B_y \partial_y g^+) \quad (115)$$

where A, B_x, B_y are fixed $d \times d$ matrices and $g(r, t) \in SU(d)$ is slowly varying in space and time, in perfect analogy with non-collinear antiferromagnets [51].

The connection between this slowly varying matrix field $g(r, t)$ and the actual time dependent spinor $\psi_a(r, t)$ is expected to be something like:

$$\psi_a(r, t) = \sum_{b=1}^d g_{ab}(r, t) \theta_{b-1}(r) \quad (116)$$

The problem that we face here is that this effective model is supposed to capture the low-energy dynamics in the classical manifold of *analytic* textures. The previous relation would force $g(r, t)$ to be holomorphic, that is $\partial_{\bar{z}}g = 0$, but then it would run into conflict with the unitarity of $g(r, t)$. We may speculate that a similar effective theory as the one of eq. (115) could continue to hold but that eq. (116) would have to be replaced by something compatible with the analyticity constraint. A natural idea would be to write:

$$\psi_a(r, t) = \mathcal{P}_{\text{hol}} \left(\sum_{b=1}^d g_{ab}(r, t) \theta_{b-1}(r) \right) \quad (117)$$

where $\mathcal{P}_{\text{hol}}$ is a projector (to be precisely defined) on the space of holomorphic functions. Note that we may have a similar deformation of the algebra of functions over the plane as the one considered in section 2.2, given the strong resemblance between the lowest Landau level and the subspace of holomorphic functions. In particular, a notion of star product should emerge, through the relation

$$\mathcal{P}_{\text{hol}}(f \mathcal{P}_{\text{hol}}(g\theta)) = \mathcal{P}_{\text{hol}}((f \star g)\theta) \quad (118)$$

These remain open questions at the present time. Even if this effective description of the residual interaction between the $d^2 - 1$ Goldstone branches in terms of a non-linear σ -model on a “non-commutative plane” is validated by future studies, another framework is likely to be needed to account for the interactions between the magneto-phonon and these Goldstone modes. So the subject is far from being closed !

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5 Appendix

5.1 Coherent states in the lowest Landau level

Let us choose the circular gauge. It is well known that the lowest Landau level corresponds in this gauge to wave functions of the form $\psi(r) = f(z) \exp(-|z|^2/(4l^2))$, where $z = x + iy$ if $r = (x, y)$. An orthonormal basis on the infinite plane is given by:

$$\langle r | n \rangle = \frac{(z/l)^n}{\sqrt{2\pi 2^n n!} l} e^{-\frac{|z|^2}{4l^2}} \quad (119)$$

The projector on the lowest Landau level is then constructed as:

$$\mathcal{P}_{\text{LLL}} = \sum_{n=0}^{\infty} |n\rangle\langle n| \quad (120)$$

Explicitely;

$$\langle r|\mathcal{P}_{\text{LLL}}|r'\rangle = \frac{1}{2\pi l^2} \exp\left(\frac{z\bar{z}'}{2l^2} - \frac{|z|^2 + |z'|^2}{4l^2}\right) \quad (121)$$

The state $\mathcal{P}_{\text{LLL}}|r'\rangle$ belongs to the lowest Landau level. It has the distinctive property that it is orthogonal to the codimension 1 subspace of all the wave-functions ψ which vanish at r' . Therefore, we expect it to be strongly localized around r' , which is confirmed by a direct inspection of the behavior of $|\langle r|\mathcal{P}_{\text{LLL}}|r'\rangle|$ when r gets close to r' . This is then a natural candidate to define the coherent state centered at r' . Let us normalize this state. We have:

$$\langle r'|\mathcal{P}_{\text{LLL}}^2|r'\rangle = \langle r'|\mathcal{P}_{\text{LLL}}|r'\rangle = \frac{1}{2\pi l^2} \quad (122)$$

The normalized coherent state centered at r' is thus $|\Phi_{\bar{z}'}\rangle = \sqrt{2\pi}l\mathcal{P}_{\text{LLL}}|r'\rangle$, and we have:

$$\langle r|\Phi_{\bar{z}'}\rangle = \Phi_{\bar{z}'}(r) = \frac{1}{\sqrt{2\pi}l} \exp\left(\frac{z\bar{z}'}{2l^2} - \frac{|z|^2 + |z'|^2}{4l^2}\right) \quad (123)$$

This expression motivates the notation because r' is involved only through $\bar{z}'$ in the analytical part (i.e. function of z) of the wave-function $\Phi_{\bar{z}'}(r)$. The overlap between two coherent states is given by:

$$\langle\Phi_{\bar{z}}|\Phi_{\bar{z}'}\rangle = \exp\left(\frac{z\bar{z}'}{2l^2} - \frac{|z|^2 + |z'|^2}{4l^2}\right) \quad (124)$$

In particular, since $|\langle\Phi_{\bar{z}}|\Phi_{\bar{z}'}\rangle| = \exp(-|z - z'|^2/4l^2)$, this overlap decreases very quickly beyond a spatial scale given by the magnetic length. We also see that when this length goes to zero, coherent states become orthogonal, which fits nicely with our intuition of the classical limit.

5.2 From covariant symbols on a 2D plane to operators

It is convenient to introduce a raising operator b^+ and a lowering operator b , both acting in the lowest Landau level according to:

$$b^+|n\rangle = \sqrt{n+1}|n+1\rangle, \quad b|n\rangle = \sqrt{n}|n-1\rangle \quad (125)$$

They satisfy the usual commutation relation $[b, b^+] = 1$. From this definition and the expression (123) for coherent states, we see that:

$$|\Phi_{\bar{z}}\rangle = e^{-\frac{|z|^2}{4l^2}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \left(\frac{\bar{z}}{\sqrt{2}l}\right)^n |n\rangle = e^{-\frac{|z|^2}{4l^2}} \exp\left(\frac{\bar{z}}{\sqrt{2}l}b^+\right)|0\rangle \quad (126)$$

From this, we check the important relations:

$$b|\Phi_{\bar{z}}\rangle = \frac{\bar{z}}{\sqrt{2}l}|\Phi_{\bar{z}}\rangle, \quad \langle\Phi_{\bar{z}}|b^+ = \frac{z}{\sqrt{2}l}\langle\Phi_{\bar{z}}| \quad (127)$$

This shows that the covariant symbol is very easy to extract for *normal-ordered* operators (or *Wick-ordered* operators in Berezin's terminology), i.e. linear combinations of monomials of the form $(b^+)^m b^n$. The previous relations show indeed that:

$$\widehat{z^m \bar{z}^n} = (\sqrt{2}l)^{m+n} (b^+)^m b^n \quad (128)$$

5.3 Single particle density matrix in a texture Slater determinant

In a large magnetic field, and for a filling factor less than d , a very good approximation to study the low energy physics is to restrict the single particle states to the lowest Landau level. It is then convenient to introduce projected creation operators $\Psi_a^+(r)$ and annihilation operators $\Psi_a(r)$ for a single electron in internal state a at position r . Using the orthogonal basis $|n\rangle$ for $n = 0, 1, \dots$ these operators are given by:

$$\Psi_a(r) = \sum_{n=0}^{\infty} \langle r|n\rangle c_{a,n}, \quad \Psi_a^+(r) = \sum_{n=0}^{\infty} \langle n|r\rangle c_{a,n}^+ \quad (129)$$

Here $c_{a,n}$ destroys an electron in the internal state a and the basis orbital state $|n\rangle$, and $c_{a,n}^+$ is the corresponding creation operator. Assuming the canonical anticommutation rules $\{c_{a,n}, c_{a',n'}^+\} = \delta_{aa'} \delta_{nn'}$, we have the less conventional:

$$\{\Psi_a(r), \Psi_{a'}^+(r')\} = \delta_{aa'} \langle r|\mathcal{P}_{\text{LLL}}|r'\rangle \quad (130)$$

The effect of the projection is to smear the expected delta function $\delta(r - r')$ on the right-hand side and to replace it by the matrix elements of the projector $\mathcal{P}_{\text{LLL}}$. Let us denote, as in the main text, by $\hat{P}_\psi$ the projector on the occupied subspace in the Slater determinant associated to the classical texture $\psi_a(r)$, and let $P_\psi(z, \bar{z}) \equiv P_\psi(r)$ be the corresponding covariant matrix symbol. A very useful fact is that we can express the single particle density matrix in this Slater determinant, in terms of the covariant symbol $P_\psi(r)$. Indeed:

$$\langle \Psi_b^+(r') \Psi_a(r) \rangle = \langle r|\mathcal{P}_{\text{LLL}}(\hat{P}_\psi)_{ab} \mathcal{P}_{\text{LLL}}|r'\rangle \quad (131)$$

So:

$$\langle \Psi_b^+(r') \Psi_a(r) \rangle = \frac{1}{2\pi l^2} \langle \Phi_{\bar{z}} | (\hat{P}_\psi)_{ab} | \Phi_{\bar{z}'} \rangle = \frac{1}{2\pi l^2} (P_\psi)_{ab}(r, r') \quad (132)$$

The last term has to be understood as the result of a continuation process from the diagonal part $(P_\psi)_{ab}(r)$ to the non-diagonal one $(P_\psi)_{ab}(r, r')$. For any operator $\hat{A}$ acting on the lowest Landau level, with covariant symbol $A(z, \bar{z}) = A(r)$, we have the useful relation:

$$A(r, r') = A\left(\frac{r+r'}{2} + \frac{i}{2}\hat{z} \times (r' - r)\right) \exp\left(-i\frac{r \times r'}{2l^2} - \frac{(r-r')^2}{4l^2}\right) \quad (133)$$

Finally, our main result is:

$$\langle \Psi_b^+(r') \Psi_a(r) \rangle = \frac{1}{2\pi l^2} (P_\psi)_{ab} \left(\frac{r+r'}{2} + \frac{i}{2}\hat{z} \times (r' - r)\right) \exp\left(-i\frac{r \times r'}{2l^2} - \frac{(r-r')^2}{4l^2}\right) \quad (134)$$

To prove eq. (133), let us consider the operator $\hat{f}_{\lambda,\mu}$ whose covariant symbol is $f_{\lambda,\mu}(r) \equiv f_{\lambda,\mu}(z, \bar{z}) = \exp(\lambda z + \mu \bar{z})$. By taking derivatives with respect to the external variables λ and μ , we can reach any monomial in z and $\bar{z}$. From eq. (128), we see that:

$$\hat{f}_{\lambda,\mu} = \exp(\sqrt{2}l\lambda b^+) \exp(\sqrt{2}l\mu b) \quad (135)$$

The extended symbol is then easily computed:

$$f_{\lambda,\mu}(r, r') \equiv \langle \Phi_{\bar{z}} | \hat{f}_{\lambda,\mu} | \Phi_{\bar{z}'} \rangle = \exp(\lambda z + \mu \bar{z}') \langle \Phi_{\bar{z}} | \Phi_{\bar{z}'} \rangle \quad (136)$$

Now if an analytic function of x and y , $g(x, y) = g(z, \bar{z})$ is continued to form the function $g(z, \bar{z}')$, the variable $x = (z + \bar{z})/2$ has to be changed into $x_{\text{new}} = (z + \bar{z}')/2$. Likewise, $y = i(\bar{z} - z)/2$ has to be changed into $y_{\text{new}} = i(\bar{z}' - z)/2$. Expliciting, we get:

$$\begin{aligned} x_{\text{new}} &= \frac{1}{2}(x + x') - \frac{i}{2}(y' - y) \\ y_{\text{new}} &= \frac{1}{2}(y + y') + \frac{i}{2}(x' - x) \end{aligned}$$

This is conveniently expressed as:

$$r_{\text{new}} = \frac{1}{2}(r + r') + \frac{i}{2}\hat{z} \times (r' - r) \quad (137)$$

Using the expression (124) for the overlap between coherent states, we see that:

$$f_{\lambda,\mu}(r, r') = f_{\lambda,\mu} \left(\frac{1}{2}(r + r') + \frac{i}{2}\hat{z} \times (r' - r) \right) \exp \left(-i \frac{r \times r'}{2l^2} - \frac{(r - r')^2}{4l^2} \right) \quad (138)$$

which establishes eq. (133).

5.4 Hamiltonians with quadratic covariant symbol

To keep the discussion simple, let us consider a single degree of freedom, whose associated phase-space is the plane. It is convenient to use the notations of sections 5.1 and 5.2, with the replacement $l^2 = \hbar$. Suppose that for some Hamiltonian $\hat{H}$, the covariant symbol $H(z, \bar{z}) \equiv \langle \Phi_{\bar{z}} | \hat{H} | \Phi_{\bar{z}} \rangle$ is minimal at some point z_0 . By a translation, it is possible to shift the origin in the plane, so we may assume that $z_0 = 0$. The Taylor expansion of $H(z, \bar{z})$ around the origin contains no linear term, so it reads:

$$H(z, \bar{z}) = E_0 + \frac{\omega_0}{2}\bar{z}z + \frac{\Delta}{4}z^2 + \frac{\bar{\Delta}}{4}\bar{z}^2 + \dots \quad (139)$$

Here ω_0 is real and positive, and Δ is complex. The corresponding quantum Hamiltonian $\hat{H}$ can be written in terms of b and b^+ operators using the normal ordering prescription, as shown in section 5.2 above:

$$\hat{H} = E_0 + \hbar\omega_0 b^+b + \frac{\hbar\Delta}{2}(b^+)^2 + \frac{\hbar\bar{\Delta}}{2}b^2 + \dots \quad (140)$$

After a $U(1)$ rotation $b \rightarrow be^{i\alpha}$, $b^+ \rightarrow b^+e^{-i\alpha}$, it is possible to eliminate the phase of Δ , which will then be assumed to be real. This quadratic Hamiltonian is diagonalized by

a Bogoliubov transformation: $b = \cosh \theta \beta - \sinh \theta \beta^+$. Terms of the form β^2 or $(\beta^+)^2$ disappear if we choose $\tanh(2\theta) = \Delta/\omega_0$. We get then:

$$\hat{H} = E_0 + \frac{\hbar}{2}(\Omega - \omega_0) + \hbar\Omega b^+b \quad (141)$$

Here $\Omega = \sqrt{\omega_0^2 - \Delta^2}$ is the oscillation frequency of the harmonic mode. The quantum zero point correction to the variational estimate E_0 of the ground-state energy is:

$$H_{\text{qzpc}} = \frac{\hbar}{2}(\Omega - \omega_0) \quad (142)$$

This correction is always negative and it vanishes only when $\Delta = 0$. This is in agreement with the fact that the normal-ordered term $\hbar\omega_0 b^+b$ is diagonal in the standard oscillator basis and gives zero when applied to the coherent state $|\Phi_0\rangle = |0\rangle$. Only the $(b^+)^2$ term and its hermitian conjugate can change the ground-state energy, but it is purely off-diagonal, so the corresponding energy shift has to be negative.

This discussion may seem surprising to experienced readers who expect $\hbar\Omega/2$ to be the quantum zero point energy of a harmonic oscillator. In fact, there is no contradiction between both viewpoints, which differ only in a different choice of the reference energy. If we start from the traditional harmonic oscillator Hamiltonian $\hat{H} = \omega_0(\hat{p}^2 + \hat{q}^2)/2$, we get $\hat{H} = \hbar\omega_0 b^+b + \hbar\omega_0/2$. The corresponding covariant symbol is $H(z, \bar{z}) = E_0 + \omega_0|z|^2/2$, so the traditional zero point energy is incorporated in the constant $E_0 = \hbar\omega_0/2$. In physical terms, H_{qzpc} measures the additional energy shift due to squeezing phenomena induced by non-diagonal operators such as $(b^+)^2$ and b^2 . If the coherent states initially chosen in our variational procedure are the correct ones, there are no such squeezing terms, and the traditional zero point energy is already correctly taken into account by computing the expectation value of the quantum mechanical operator $\hat{H}$ in the chosen coherent state family. In our approach, a non-zero H_{qzpc} signals that residual quantum fluctuations will dress the initial optimal coherent state and introduce a finite amount of squeezing. Most of the above remarks can be generalized to quadratic Hamiltonians with an arbitrary finite number of oscillators, but to keep the discussion simple we won't elaborate further on this issue.

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