Interaction effects in topological insulators - New Phases
Lecture 1

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1 Lecture 1

1.1 Quantum Phases of Matter. Short vs. Long Range Entanglement

How do we distinguish different phases of matter? We will be particularly interested in zero temperature states - i.e. the ground state of an interacting bunch of particles. Typically, the phases of matter are only sharply defined in the limit of an infinite number of particles. Then, two states belong to different phases, if there is necessarily a phase transition separating them, where properties change in a singular fashion. For a while, people thought they had figured out how to diagnose this. The answer they believed had to do with symmetry - at the fundamental level, breaking symmetry in different ways lead to different phases. For example - in the Quantum Ising model, with two level systems arranged on a line:

$$H = -J \sum_i \left(\sigma_i^z \sigma_j^z + g \sigma_i^x \right) \tag{1}$$

there is a symmetry or flipping the spin $\sigma^z \to -\sigma^z$ (similarly for the y spin direction). This Z_2 symmetry is spontaneously broken if g is small while it is restored if g is sufficiently large. Thus there are two phases which can be distinguished by the order parameter $\langle \sigma_i^z \rangle$. Symmetry is key to having a sharp distinction - if it is broken by hand, eg. by adding a field along σ^z, then the phase transition can be converted into a crossover. For a while it was thought that all phases of matter (apart from a few well characterized outliers like metals) could be identified by such a procedure.

However, Wegner cooked up a model which could be shown to have two phases which shared the same symmetry. Today we understand that they differ in their topology - here is the modern avatar of that model, the Kitaev Toric

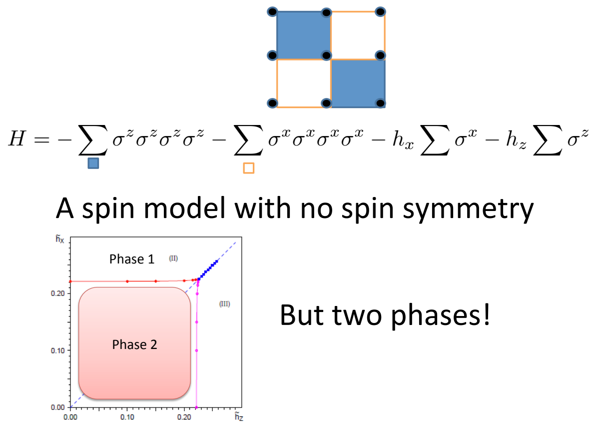

Figure 1: The toric code model with generic perturbations, which has two phases although they have the same symmetry. The Phase 2 is gapped but has long range entanglement - as evidenced by having ground state degeneracy with periodic boundary conditions, and anyone excitations with nontrivial mutual statistics.

code, which also has two level systems on the vertices of a 2D square lattice. The coupling takes the following form - and includes 4 spin couplings around the plaquettes (see figure 1).

$$H = -\sum_{black} \sigma_i^z\sigma_j^z\sigma_k^z\sigma_l^z - \sum_{white} \sigma_i^x\sigma_j^x\sigma_k^x\sigma_l^x - h_z \sum_i \sigma_i^z - h_x \sum_i \sigma_i^x \tag{2}$$

The two phases in this model include a ‘trivial’ phase which can be thought of as a product state of spins pouting along a certain direction. The other phase does not have any representation as a product state of a site or finite group of sites. It can be thought of as a condensate of closed loops - where the loops are found by linking $\sigma^z = -1$ spins for example. There are two kinds of point excitations in this phase - which violate the individual plaquette terms. One is called a ‘charge’ and the other the ‘vortex’. Despite ultimately being built out of bosons (the spins can equally be thought of as occupying sites with hard core bosons), the excitations have unusual statistics. Taking one around mother leads to a (-1) sign, hence they are mutual semions. This is an example of fractional statistics, in the generalized sense, which includes both exchange of identical particles as well as mutual statistics. This is an indication of long range entanglement (LRE). Another signature is that when the system is defined with periodic boundary conditions (i.e. on a torus), there is a ground state degeneracy. The degenerate states appear identical with respect to any local operator (the degeneracy itself can be understood since the Hamiltonian

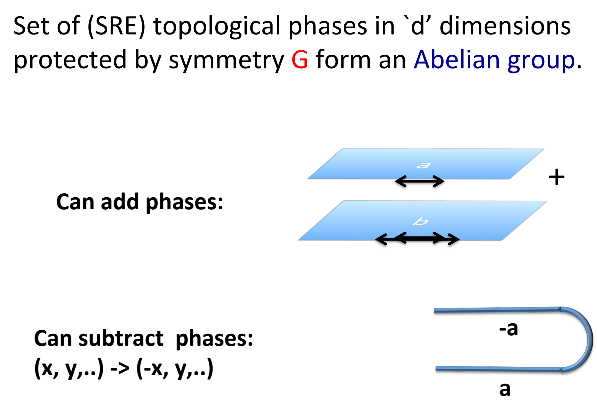

Figure 2: SRE phases that preserve a symmetry must form an Abelian group. This is not true for LRE phases, which typically get more complicated on combining them together.

is a local operator). We define a short range entangled phase as one that does *not* have these properties.

A Short Range Entangled (SRE) state is a gapped phase with a unique ground state on a closed manifold. All excitations (particles with short range interactions) have conventional statistics. For example, if the phase is built of bosons, all excitations are bosonic with trivial mutual statistics.

We also allow for the possibility of a symmetry specified by the group G. We will restrict attention to *internal symmetries* , that is, we do not consider symmetries that change spatial coordinatees, like inversion, reflection, translation etc. Common internal symmetries that are encountered in condensed matter physics are charge conservation, various types of spin rotation symmetry, and time reversal symmetry. The advantage of working with internal symmetries is that we can consider disordered systems that respect the symmetry. also the symmetries can be defined at the edge, while for spatial symmetries, one may require a special edge configuration, to preserve symmetry. Some spatial symmetries like inversion are always broken at the edge.

Gapped SRE ground states that preserve their internal symmetries only differ from the trivial phase if they possess edge states. (For 1D systems, the edge states are always gapless excitations, and rigorous statements can be made using matrix product state representation of gapped phases.)

The fact that SRE topological phases only differ at the edge, not in the bulk (unlike LRE state), makes them much easier to study. The set of SRE topological phases in a given dimension with symmetry G actually has more structure than just a set. If we add the trivial phase as an 'identity' element, the set of phases actually form an Abelian group. The operations for the group are shown. The addition operation is obvious: take two states and put them side-by-side. But it is not so obvious that a state has an inverse: how is it possible to cancel out edge states? Two copies of a topological insulator cancel one another, because the Dirac points can be coupled by a scattering term that makes a gap. The inverse of a phase is its mirror image (i.e reverse one of the coordinates). To see this, we must show that the state and its mirror image cancel; the argument is illustrated at the bottom of figure 2. In one dimension, for example, take a closed loop of the state, and flatten it. The ends are really part of the bulk of the loop before it was squashed, so they are gapped. Therefore this state has no edge states. Because topological SRE phases are classified by their edge states, it must be the trivial state. Therefore the Hamiltonian describing the inverse of a particular state $H(x,\, y,\, z\ldots)$ is, for example $H(-x, y, z\ldots)$.

1.2 Examples of SRE Topological phases

Let us give a couple of concrete examples of SRE topological phases of bosons/spins. These are necessarily interacting - unlike free fermions, there is no 'band' picture here.

1.2.1 Haldane phase of S=1 antiferromagnet in d=1

The following simple Hamiltonian actually leads to a gapped phase with SRE, but gapless edge states:

$$H = J \sum_i \vec{S}_i \cdot \vec{S}_{i+1}$$

The edge realizes effectively a $S = 1/2$ state, despite the chain being built of $S = 1$ spins. This phenomena has been observed experimentally in some nickel based insulating magnets, e.g. Y_2BaNiO_5. The Ni atoms form S=1 spins, organized into chain like structures that are relatively well isolated from one another .

The symmetry that is crucial to protecting this phase is SO(3) spin rotation symmetry. However, it turns putt hat the full rotation symmetry is not required. It is sufficient to just retain the 180 degree rotations about the x, y and z axes. This symmetry group $\{I,\, X,\, Y,\, Z\}$, contains the identity and the 3 rotation elements. This can be written as $\{I,\, X\} \times \{1,\, Y\}$, since $Z = X \times Y$, the combination of two rotations is the third rotation. Mathematically this group is $Z_2 \times Z_2$. We will write down a model with this symmetry (which is not quite reducing the S=1 down to this rotation symmetry), but which has the advantage of being exactly soluble - not just for the ground state but also for all the excited states. This model also has a nice interpretation - of arising from condensing domain walls bound to spin flips.

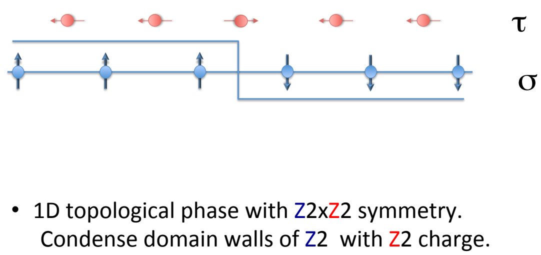

Figure 3: An exactly soluble model of a 1D SRE phase with gapless edge states protected by $Z_2 \times Z_2$ symmetry. Terms in this Hamiltonian encourage binding of domain walls to spin flips. The topological phase emerges on condensing these 'decorated' domain walls.

1.2.2 An exactly soluble topological phase in d=1

Consider a spin model with $Z_2 \times Z_2$ symmetry. There is a Z_2 set of topological phases with this symmetry in d=1, and we will explicitly construct the nontrivial topological phase. We will implement this symmetry by a pair of Ising models (labeled σ and τ) that live on a zig-zag lattice as shown in the Figure. Consider beginning in the ordered state of the σ, but where the τ are disordered and point along a transverse field τ^x. Now, we would like to restore the Z_2 symmetry of the σ spins. We do this by condensing the domain walls of the σ spins. If we directly condense domain walls we get the trivial symmetric state. However, we can choose to condense domain walls with a spin flip of τ attached. We will see that this gives the topological phase.

One way to do this is to write down a Hamiltonian that would lead to this binding. Note that the operator $\sigma_i^z \sigma_{i+1}^z$ detects a domain wall. Consider:

$$H = -\sum_i \left(\sigma_{2i}^z \tau_{2i+1}^x \sigma_{2i+2}^z + \tau_{2i-1}^z \sigma_{2i}^x \tau_{2i+1}^z \right) \tag{3}$$

where we have placed the σ (τ) on the even (odd) sites of the lattice. In the absence of a domain wall, we have the usual transverse field term, whose sign changes when a domain wall is encountered. We will show that this is a gapped phase with short range entanglement, but has gapless edge states.

First, consider the system with periodic boundary conditions. We will leave it as an exercise to show that each of the terms in the Hamiltonian 3 commutes with all others. Then, for a system with N sites, we have exactly N terms which can be written as $H = -\sum_i \left(\tilde{\sigma}^x_{2i} + \tilde{\tau}^x_{2i+1}\right)$, where the tilde denote the three spin operators in the Hamiltonian 3. Hence, this simply looks like each site has a modified transverse field, which implies a unique ground state and a gap, in this system with periodic boundary conditions.

Now consider open boundary conditions as shown. Let us focus on the left edge, where the end of the chain implies we lose $\tilde{\sigma}_x$ operator. This will result in a two fold degeneracy as we will show. The first term in the Hamiltonian is now $-\sigma^z_0\tau^x_1\sigma^z_2$. We can easily show that the following two operators commute with the Hamiltonian $\Sigma^z = \sigma^z_0$ and $\Sigma^x = \sigma^x_0\tau^z_1$. However they anti commute with one another. Hence we can show the ground state must be at least two fold degenerate. Say you had a unique ground state of the Hamiltonian, $|\psi\rangle$. This must be an eigenstate of Σ^z, since it commutes with the Hamiltonian. Let us say $\Sigma^z|\psi\rangle = \lambda|\psi\rangle$, where $\lambda = \pm 1$. However, we can find an independent state $|\psi'\rangle = \Sigma^x|\psi\rangle$. This is a degenerate state since $[\Sigma^x, H] = 0$. It is also a distinct state since it has a different eigenvalue $\Sigma^z|\psi\rangle = -\lambda|\psi\rangle$, due to $\Sigma^z\Sigma^x = -\Sigma^x\Sigma^z$. Hence there are at least two ground states $(|\psi\rangle, |\psi'\rangle)$. They only differ by application of an edge operator, hence this is an edge degeneracy.

Note, it is important we preserve the symmetry - if we add Σ^a to the Hamiltonian we can gap the edge state, but at the expense of also breaking the $Z_2 \times Z_2$ symmetry. Hence this is called a symmetry protected topological phase (SPT). This model has special properties that make it exactly soluble - but adding general perturbations that are local and preserve the symmetry lead to a more generic state. The presence of an energy gap implies that the state is stable against weak perturbation, which means it will remain in the same phase.

Exercises:

1. Verify that the terms in Eqn. 3 commute with one another, in a chain with periodic boundary conditions. With open boundaries, explicitly write out a Hamiltonian and check that the edge operators Σ^a commutes with it.

2. Use the Jordan Wigner procedure to map Eqn. 3 onto fermion operators. Recall, for the 1D quantum Ising model, the transformation is $c^\dagger_j = \sigma^+_j S_j$ where the string operator is $S_j = \prod_{i>j} \sigma^x_i$. Show that the same mapping leads to a topological phase of these nonocal fermions. Note however there are some important differences with a topological phase of electrons. Argue that in the latter case there must always be a residual symmetry that cannot be broken by any physical operator, unlike in the fermionized version of the problem above.

References Some of this material can be found in a review (Ari Turner and A.V. `http://xxx.lanl.gov/pdf/1301.0330.pdf)`