Interaction effects in topological insulators - New Phases
Lecture 2 (Provisional Notes)

Ashvin Vishwanath
UC Berkeley

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1 Lecture 2

Let us give a couple of concrete examples of SRE topological phases of bosons/spins. These are necessarily interacting - unlike free fermions, there is no 'band' picture here. Later, we will see how they combine with the free fermion topological phases, in particular the Integer Quantum Hall effect, to extend the classification from a single integer to a pair of integers.

1.1 SRE phase of bosons in d=2

Let us consider a system of two species of bosons (A and B say these are two species of atoms in an optical lattice), whose numbers are individually conserved. By analogy with the 1D example, we want to find a disordered phase that respects these symmetries, and can be obtained by condensing a vortex combined with a charge. In particular, say we begin in the superfluid state of the 'A' bosons. We want to exit it by condensing vortices (and anti vortices). However, in order to avoid reaching the regular Mott insulator, we will bind a +1 charge of 'B' boson to this vortex (and -1 charge on the anti vortex) before condensing them. We will show that this gives rise to a SRE topological phase. The phase will have gapless edge state protected by symmetry. We will implement this in two ways (i) by a coupled wire construction and (ii) by writing down an effective field theory.

1.1.1 Coupled Wire Construction

Consider an array of wires indexed by i. Species A (B) bosons are placed on the even (odd) wires. Hence they only hop between even or between odd sites. The field theory for a single chain takes the form

$$H = J(\partial_x\phi)^2 + U(\partial_x\theta)^2 \tag{1}$$

where ϕ is the boson phase and θ is defined as

$$\partial_x \theta(x) = 2\pi n(x)$$

where n is the particle density. Using the standard density-phase commutation relation we have $[\partial_x \theta(x), \phi(x')] = -2\pi i \delta(x - x')$. This can be integrated to give:

$$[\theta(x), \phi(x')] = -\pi i \mathrm{Sign}(x - x') \tag{2}$$

where $\mathrm{Sign}(x) = \pm 1$ for $x > 0$ ($x < 0$).

Exercise: Show that this commutation relation is symmetric on interchanging the fields $\theta \leftrightarrow \phi$.

Normally, when the bosons are at commensurate filling one also has a vortex tunneling operators $H_v = -\sum_n \lambda_n \cos(n\theta)$. When these are relevant, a Mott insulator results - for example when $-\cos\theta$ is large, the field is pinned at $\theta = 0$, which implies that the density is uniform as in a Mott state. An excitation is a soliton, that is $\theta(x \ll 0) \to 0$ while $\theta(x \gg 0) \to 2\pi$. This costs a finite energy which is the gap to particle excitations in the Mott state.

However, here we will be interested in a different type of vortex condensate, one that binds charge. To this end consider building a 2D system from coupling 1D systems (tubes) together. We will assume that on alternate tubes we have bosons of species the two different species, thus on the even numbered tubes $2i$ we have A and on the odd $2i + 1$ we have B. In the absence of interchain coupling w shave the decoupled Luttinger Liquid Hamiltonian:

$$H_0 = K \sum_i \left[(\partial_x \theta_i)^2 + (\partial_x \phi_i)^2 \right] \tag{3}$$

and

$$[\theta_j(x), \phi_k(x')] = -i\pi \delta_{j,k} \delta(x - x') \tag{4}$$

We ignore the effect of conventional vortices. Let us instead attempt to condense vortices of the 'A' bosons with B charge and vice versa. We will do this by allowing the composite objects to 'hop', which will lower their energy and make them eventually 'Bose condense'. Of course, since this is a vortex condensate we are led to an insulator as elaborated previously. We will see that this is an exotic insulator.

Note, the vortices of the 'A' (blue) bosons naturally live in the centers of plaquettes of the sites available to 'A'. This happens to be on the 'B' tubes as shown by the blue cross in the figure. Hence, if we hop a vortex from one of the blue crosses to the adjacent one, this can be represented as a space-time event on tube $i = 2$, represented as $e^{i\theta_2}$. However, we also simultaneously want to hop bosons 'B' and we have arranged for their lattices sites to coincide with the location of the 'B' vortices. This combined process is then written as: $e^{i(-\phi_1+\phi_3+\theta_2)}$. The reverse process binds an anti-vortex to a 'hole': $e^{i(+\phi_1-\phi_3-\theta_2)}$ and taken to gather this leads to the set of terms:

$$H_{int} = -\lambda \sum_i \cos(\phi_{i-1} - \theta_i - \phi_{i+1}) \tag{5}$$

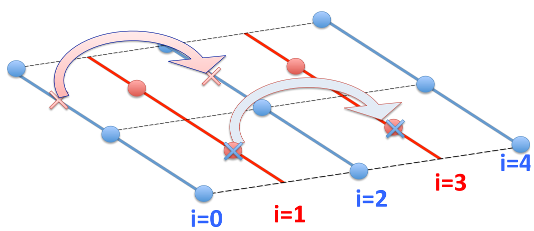

Figure 1: Coupled wire construction of bosonic topological phase. ‘A’ (‘B’) bosons, that live on the even (odd) numbered tubes. Vortices of one species are bound to bosons of the opposite species and their tunneling is shown by the arrows. Thes processes are shown to commute and gap the bulk, but leave behind an topological edge state.

First we note that these terms all commute with one another, for example, if we denote $\tilde{\theta}_i = \theta_i + (\phi_{i+1} - \phi_{i-1})$, then any two terms commute:

$$\left[\tilde{\theta}_i(x),\, \tilde{\theta}_j\Big(x'\right)] = 0$$

Exercise: Calculate the commutator of a pair of fields $\Phi_{l,m}$ and $\Phi_{l',m'}$ where $\Phi_{l,m} = \sum_i (l_i\phi_i + m_i\theta_i)$. Use this to prove the result above.

Thus all of these can be simultaneously satisfied. If we have periodic boundary conditions, then there is a unique ground state rather like pinning the θ fields gives a unique Mott insulating state. The same count of variables leads to a unique state in this case. However, interesting edge states appear if we have an open slab as in the Figure 1. Let us focus on the left edge. Clearly we are missing a cosine pinning field for $\tilde{\theta}_0$. This means that the first non vanishing cosine terms are $\tilde{\theta}_1 = -\phi_0 + \theta_1 + \phi_2$ and $\tilde{\theta}_2 = -\phi_1 + \theta_2 + \phi_3$. An field that commutes with both these is $\Phi = \phi_0$. However, there is also a conjugate field we require to define the edge dynamics. In an isolated chain this would have been θ_0 However, here this does not commute with one of the first cosine. It can be rectified by adding $\Theta = \theta_0 + \phi_1$, which commute with all the cosines, and has the standard Luttinger liquid commutator with Φ: $[\Theta(x),\, \Phi(x')] = -i\pi\delta(x - x')$. Hence we have a gapless edge mode - which is described by the usual Luttinger liquid theory. However, there is an important difference between this Luttinger liquid at the edge and one that can be realized in purely 1D. It is impossible to gap this edge without breaking one of the symmetries, which happens to be number conservation of the two boson species. This is an internal symmetry - and in a purely 1D system it is always possible to find a gapped state that preserves all internal symmetries - we just combine degrees of freedom till they transform in a trivial way under the symmetry and condense them. However, this is not possible at the edge - one cannot condense Φ since it is charged under the U(1) symmetry that protects 'A' particle conservation, and similarly we cannot introduce a cosine of the Θ field since it is charged under the other U(1). This is an indication that it is a topological phases - we will see this also implies a quantized Hall conductance.

1.1.2 Effective Field Theory

Let us write down an effective theory to describe a fluid built out of boson-vortex composites. The 'A' particles acquire a phase of 2π on circling vortices, hence the effect go vortices can be modeled by a vector potential whose curl is centered at the vortex locations: $\partial_x a_y - \partial_y a_x = 2\pi \sum_j n_j^v \delta(r - r_j^v)$, where $(n_j^v,\, r_j^v)$ represent the strength and location of the vortices. This vector potential will couple minimally to the current $\mathcal{L} = \vec{j}_A \cdot \vec{a}$, where the vectors are two-vectors. A rewriting o this formalism to include motion of vortices results in the following generalization to the three current $j^\mu = (\rho,\, j_x,\, j_y)$, and three gauge potential $a_\mu = (a_0,\, a_x,\, a_y)$. Also, since we assume the vortices are bound to the bosons 'B' we can rewrite the equation for a as:

$$\epsilon^{\mu\nu\lambda}\partial_\nu a^B_\lambda = 2\pi j^\mu_B \tag{6}$$

where we have introduced a superscript 'B' for the vector potential. At the same time we can utilize the continuity equation for the current j_A, $\partial_\mu j^\mu_A = 0$ to write:

$$\epsilon^{\mu\nu\lambda}\partial_\nu a^A_\lambda = 2\pi j^\mu_A \tag{7}$$

In order to keep track of the charge density of 'A' and 'B' bosons, it is useful to introduce external vector potentials $A^{(A,\,B)}$ that couple to the currents of the 'A' and 'B' bosons. This leads to our final topological Lagrangian:

$$\begin{aligned}\mathcal{L}_{topo} &= \frac{\epsilon^{\mu\nu\lambda}}{2\pi}\left(a^A_\mu\partial_\nu a^B_\lambda + A^A_\mu\partial_\nu a^A_\lambda + A^B_\mu\partial_\nu a^B_\lambda\right) &(8)\\ \mathcal{Z}_{topo}[A^A, A^B] &= e^{iS_{topo}} = \int \mathcal{D}a^A\mathcal{D}a^B e^{i\int dxdydt\,\mathcal{L}_{topo}} &(9)\end{aligned}$$

First, we would like to establish that this describes a phase with short range entanglement. Note, the mutual phases involved are all 2π implying the absence of fractional statistics. One can also compute the ground state degeneracy on the torus - this turns out to be directly computable from this theory - if we write

$$\mathcal{L} = \frac{K_{IJ}}{4\pi}\epsilon^{\mu\nu\lambda}a^I_\mu\partial_\nu a^J_\lambda$$

, the ground state degeneracy is $|detK|$. In this case $K = \sigma^x$, and there is a unique ground state.

Given that it is a SRE phase, we can deduce two important consequences from this theory- the first is regarding edge states, which can be shown to be equivalent to that derived before, and the quantized Hall conductivity. The latter is obtained by integrating out the a fields to obtain an action purely in terms of the external probe fields A. The current is then defined as $j_A = \frac{\delta S}{\delta A^A}$ where $Z = e^{iS}$. A gaussian integration of Eqn. 9 yields:

$$S_{topo} = -\int dxdydt\,\frac{\epsilon^{\mu\nu\lambda}}{2\pi}A^A_\mu\partial_\nu A^B_\lambda \tag{10}$$

thus we have:

$$j^\mu_A = -\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu A^B_\lambda \tag{11}$$

If we consider the spatial components of this equation we find: $j^x_A = \frac{1}{2\pi}E^y_B$, where E_B is the electric field applied on species 'B'. Thus we have a crossed response Hall conductivity $\sigma^{AB}_{xy} = \frac{1}{2\pi}$, which , replacing charge Q_a for the bosons and $\hbar$ gives $\sigma^{AB}_{xy} = \frac{Q_AQ_B}{h}$.

We would like to apply these insights to electronic systems, where one may combine pairs of electrons to form Cooper pairs with charge $Q = 2e$. However, in that case there is a single conservation law. Topological phases with a single $U(1)$ can be described by the field theory above 9 , if we assume that the two species of bosons can tunnel into one another and collapse the combined $U(1)\times U(1)$ symmetry into a single common $U(1)$. This amounts to replacing the pair of external vector potentials by a single one and the resting topological response theory is:

$$S_{topo} = -\int dxdydt\, \frac{\epsilon^{\mu\nu\lambda}}{2\pi} A_\mu \partial_\nu A_\lambda \tag{12}$$

Now, differentiating with respect to the vector potential we get *two* contributions and hence: $j^\mu = \frac{2}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda$ which implies a Hall conductivity, in units of the boson charge $\sigma_{xy} = 2\frac{Q^2}{h}$. This is the Bosonic Integer Quantum Hall (**BIQH**) phase. Somewhat surprisingly, its Hall conductance is always an *even* integer. Potential realization of this phase in bilayer systems of bosons in the lowest Landau level with net filling $\nu = 2$ have been discussed in recent numerical work [1].

Note we have assumed commensurate filling to admit an insulator. Also, these models are not exactly soluble in the same way that the previous models were - for other approaches to construct models in this phase see [2] .

1.1.3 Implications for Interacting Quantum Hall State of Electrons:

It is well known that free fermion IQH states have a quantized Hall conductance $\sigma_{xy} = n\frac{e^2}{h}$. At the same time, they have a quantized thermal hall effect $\frac{\kappa_{xy}}{T} = c\frac{\pi^2 k_B^2}{3h}$ where $c = n$. The latter simply counts the difference between the number of right moving and left moving edge states. This equality is an expression of the Wiedemann Franz law that related thermal and electrical conductivity for weakly interacting electrons. This leads to the familiar integer classification of IQH $\mathcal{Z}$. How is this modified in the presence of interactions? We will continue to assume short range entanglement - so that fractional quantum Hall states are excluded from our discussion. It has long been known that n must remain an integer if charge is to remain unfractionalized. However, the equality $n = c$ can be modified. In fact, if we assume the electrons can combine into Cooper pairs which form the BIQH state, the latter has Hall conductance $\sigma_{xy} = 8\frac{e^2}{h}$ but $\kappa_{xy} = 0$. Thus we can have $n - c = 8m$. Indeed this implies that the classification of interacting quantum Hall states of electrons with SRE is $\mathcal{Z} \times Z$ at least. Note, this also predicts a phase where $n = 0$ but $c = 8$. This can be achieved by combining an $n = 8$ free fermion quantum Hall state with a BIQH state of Cooper pairs to cancel the electrical Hall conductance. The remaining thermal Hall conductance is $c = 8$. It can be shown that a π flux inserted in this state has trivial statistics and can be condensed - which implies that all electrons are confined into bosonic particles without disturbing the topological response of this phase. Alternately, one can show that neutral bosons with short range interaction can lead to a topological phase with chiral edge states, if they appear in multiples of eight. Indeed one can write down a multi component chern simons theory to describe this topological phase of neutral bosons, in terms of a K matrix as described in detail below.

A phase without topological order is characterized by a symmetric K matrix with $|\det \mathbf{K}| = 1$. A chiral state in $2 + 1$-D requires the signature (n_+, n_-) of

[1] See arXiv:1305.0298, arXiv:1304.5716, arXiv:1304.7553

[2] See S. Gerdatis and O. Motrunich, arXiv:1302.1436 (in particular, Appendix C)

its K matrix to satisfy that $n_+ \neq n_-$. We therefore seek a K matrix with the following properties (1) $|\det \mathbf{K}| = 1$ (2) the diagonal elements $K_{I,I}$ are all even integers so that all excitations are bosons and (3) a maximally chiral phases, where all the edge states propagate in a single direction. Then, all eigenvalues of K must have the same sign (say positive), so $\mathbf{K}$ is a positive definite symmetric unimodular matrix.

It is helpful to map the problem of finding such a $\mathbf{K}$ to the following crystallographic problem. Diagonalizing $\mathbf{K}$ and multiplying each normalized eigenvector by the square root of its eigenvalue one obtains a set of primitive lattice vectors $\mathbf{e}_I$ such that $K_{IJ} = \mathbf{e}_I \cdot \mathbf{e}_J$. The inner product of a pair of vectors $l_I \mathbf{e}_I$ and $l'_I \mathbf{e}_I$ are given by $l'_I K_{IJ} l_J$, while the volume of the unit cell is given by $[\mathrm{Det} K]^{1/2}$. The latter can be seen by writing the components of the vectors as a square matrix: $[\mathbf{k}]_{aI} = [e_I]_a$. Then $\mathrm{Det} k$ is the volume of the unit cell. However, $K_{IJ} = \sum_a k_{aI} k_{aJ} = (\mathbf{k}^{\mathbf{T}} \mathbf{k})_{IJ}$. Thus $\mathrm{Det}\mathbf{K} = [\mathrm{Det}\mathbf{k}]^2$.

Thus, for a phase without topological order, we require the volume of the lattice unit cell to be unity $[\mathrm{Det} k] = 1$ (unimodular lattice). Furthermore, for a bosonic state, we need that all lattice vectors have even length $l_I K_{IJ} l_J =$ even integer, since the K matrix has even diagonal entries (even lattice). It is known that the minimum dimension this can occur in is eight. In fact, the root lattice of the exceptional Lie group E_8 is the smallest dimensional unimodular, even lattice[3]. Such lattices only occur in dimensions that are a multiple of 8.

A specific form of the K matrix is:

$$K = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \tag{13}$$

This matrix has unit determinant and all eigenvalues are positive. It defines a topological phase of bosons without topological order, with eight chiral bosons at the edge. We will call this the E_8 state since it is related to the E_8 group.

[3] See wikipedia entry for E_8 root lattice (Gosset lattice)