Interaction effects in topological insulators - New Phenomena and Phases
Lecture 3

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1 Lecture 3

1.1 SRE phase of bosons in d=3

We would like to understand how to describe the surface states of a 3D SRE topological phase of bosons. While for free fermions, one has Dirac or Majorana surface cones, the bosonic analog is less clear, particularly since the surface is 2D and one does not have access to bosonization and other powerful tools available for the previous problem of 1D edges.

Based on our previous experience with 1D edges, we will directly consider the surface and ask how symmetry can act in an anomalous way, to produce a topological surface. The simplest example is to consider a system with $U(1)$ and $\mathcal{T}$ symmetry, where the $U(1)$ may be considered as a conserved spin component (S_z) rather than charge. This corresponds to $U(1) \times \mathcal{T}$. One option is to break the symmetry at the surface - this is a valid surface state even for a topological bulk. Say we break the $U(1)$ to get an ordered surface (which we will call a 'superfluid' since it breaks a U(1) symmetry). To restore this symmetry we would like to proliferate vortices, that can revert us to the fully symmetric state. However, for the surface of a topological bulk, there should be an obstruction to proliferating vortices. The rolling is a potential mechanism - note the vortices here preserve time reversal symmetry, that is, a vortex is mapped to a vortex under $\mathcal{T}$. This follows from the fact that our phase degree of freedom transforms like magnetic order with $\phi \to \phi + \pi$ under time reversal, so that $e^{i\phi} \to -e^{i\phi}$. The vorticity, which is defined via $\nabla \times \nabla\phi$ is invariant under this operation. Hence we can ask - how does a vortex transform under $\mathcal{T}$?

There are two physically distinct options, whether the vortex transforms as a regular, or a projective representation. In the former case there is no obstruction to condensing vortices and restoring the symmetric phase - hence this cannot represent a topological surface state. However, the vortices can also transform as a projective representation since they are nonlocal objects. On a closed surface one must make a vortex-antivortex pair. Taken together these must transform as $\mathcal{T}^2 = +1$. However, individually they can transform as $\mathcal{T}^2 = -1$, i.e. the vortex is a Kramers doublet. Denote ψ_σ as the two component vortex field $\sigma =\uparrow, \downarrow$, which transforms as $\psi_\uparrow \to \psi_\downarrow$, $\psi_\downarrow \to -\psi_\uparrow$ under time reversal . The effective Lagrangian is:

$$\mathcal{L} = |(\partial_\mu - ia_\mu)\psi_\sigma|^2 + (\partial_\mu a_\nu - \partial_\nu a_\mu)^2 + m|\psi_\sigma|^2 + \dots \tag{1}$$

where the gauge field is determined by the bosons three current

$$\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda = 2\pi \jmath^\mu \tag{2}$$

, which includes the boson charge density j^0 and current $j^{1,2}$. The vortex-gauge field coupling is intuitively rationalized from the fact that a vortex moving around a boson acquires a 2π phase. Hence, the gauge potential a that implements this satisfies: $\partial_x a_y - \partial_y a_x = 2\pi j^0$. This is one component of the equation 2 above, the other components follow from the continuity equation $\partial_\mu j^\mu = 0$.

In this dual language, when the vortices are gapped the $U(1)$ symmetry is broken, while if they are condensed the $U(1)$ symmetry is restored. The key difference between a single component vortex field, and the Kramers doublet vortex, is that in the latter case the vortex condensate always breaks time reversal symmetry. This can be seen by considering the operator $\psi_\sigma^\dagger \sigma^a_{\sigma\sigma'}\psi_\sigma = n^a$, where σ^a are Pauli matrices. Since it is a product of a vortex-antivortex pair, it is a local operator unlike an operator that insets a vortex. Ina vortex condensate this operator will acquire a nonzero expectation value. Under time reversal it is readily seen $n^a \to -n^a$, indicating that time reversal symmetry is broken. Thus, the $U(1)$ symmetry is restored at the expense of breaking $\mathcal{T}$. This is a candidate for a topological surface state.

Exercise Establish this by introducing an external 'probe' electromagnet field that couples to the bosons $\mathcal{L}_{int} = j^\mu A_\mu = A_\mu \epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda/2\pi$ and integrate out the other fields to obtain an effective action in terms of A. Consider doing this in two limits $m > 0$ ($m < 0$), where vortices are gapped (condensed) Show that when the vortices are gapped, the effective Lagrangian is $\mathcal{L}_{eff} \sim A_\perp^2$, where $A_\perp$ is the transverse part, and this represents a U(1) broken phase ('superfluid'). On the other hand, when the vortices are condensed show that $\mathcal{L}_{eff} \sim (\partial_\mu A_\nu - \partial_\nu A_\mu)^2$ (an 'insulator').

We mention two other possible surface states that this theory.

The first is the critical point $m = 0$, where symmetries are unbroken, but the surface is gapless. This is the bosonic analog of the gapless Dirac cone of fermionic topological insulators. However, since bosons are either gapped or condensed, this requires tuning a parameter to realize. This field theory (the non compact CP^1 model) appeared before in the theory of 'deconfined quantum critical points', describing a direct transition between Neel and Valence bond solid order in spin models on the square lattice. However, there the vortices transformed projectively under spatial symmetries - such as translation and rotation. Here, an internal symmetry (time reversal) is involved - which can only occur on the surface of a 3D topological phase.

1.1.1 Surface Topological Order of 3D Bosonic SRE Phases

The second possibility is to consider condensing a pair of vortices $\Phi = \epsilon_{\sigma\sigma'}\psi_\sigma(r)\psi'_\sigma(r')$, which is a Kramers singlet. This leads to a restoration of the $U(1)$ symmetry (insulator), while preserving $\mathcal{T}$. However, this is an 'exotic' insulator with topological order (excitations that fractional statistics). Note however, the bulk 3D state is still SRE, and the exotic excitations are confined to the surface. It is readily shown that the topological order is the same as that in the toric code. Note, to show this we need to identify an e and m particle which are bosons, but with π mutual statistics. The m particle is just the unpaired vortex, which remains as a gapped excitation in this phase. Additionally, we can discuss defects in the 2-vortex condensate. These are nothing but particles - however, the 2-condensate allows for a fractional particle. To see this consider the the effective 2-vortex theory $\mathcal{L}_{2v} = |(\partial_\mu - 2ia_\mu)\Phi|^2 + (\partial_\mu a_\nu - \partial_\nu a_\mu)^2 + m_2|\Phi|^2 + \ldots$, which can be obtained from (1) by considering an interaction that pairs vortices and ignoring the gapped single vortices. In the 2-vortex condensate one can consider vortices - which are obtained from the flux quantization condition $2(\partial_x a_y - \partial_y a_x) = 2\pi$, but since the flux is related to particle density, this implies a particle with charge 1/2 that of the fundamental bosons. Clearly, taking a half charge around a vortex leads to π phase. Hence this is the m particle.

This surface topological order provides a powerful way to characterize a 3D topological phase. The surfaces of SRE topological phases should be distinct from states that can be realized purely in the lower dimension. The way this works with surface topological order is that although the topological order itself can be realized in 2D, the way the excitations transform under symmetry cannot be realized in a purely 2D setup. For example here the m particle is a Kramers doublet while the e particle carries half charge of the boson (and may or may not be a Kramers doublet).

While in this case it is not immediately apparent that this is forbidden in 2D, we can give another example that arises where this is obvious. Consider the situation where both e and m particles carry half charge - this is one of the surface topological orders associated with U(1) charge and T symmetry. . We can show that this state cannot be $\mathcal{T}$ symmetric if realized in 2D, where it can be described by a K matrix CS theory:

$$\mathcal{L}_{CS} = \frac{2}{2\pi} a_1 \cdot \nabla \times a_2 - \frac{\nabla \times A}{2\pi} \cdot (a_1 + a_2) \tag{3}$$

coupling to the external A ensures that we can keep track of the charge. Note, $K = 2\sigma_x$ ensures we have toric code type Z_2 topological order (|Det K|=4). Now, integrating out a, we obtain $\mathcal{L}_{eff} = -\frac{1}{4\pi} A \cdot \nabla \times A$. This implies that if this state is realized in 2D it will have a non vanishing Hall conductance, $\sigma_{xy} = Q^2/h$ contradicting the fact that it is $\mathcal{T}$ symmetric. However it can be realized retaining $\mathcal{T}$ symmetry on the surface of a topological phase.

The simplest way to argue this is the following construction coupled layer construction, analogous to the 1D and 2D cases that we discussed before [1].

[1] See C. Wang and T. Senthil, arXiv: 1302.6234 for details

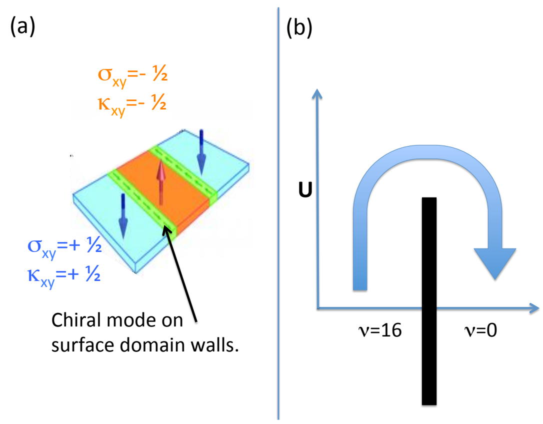

Figure 1: (a) Magnetic domains on the surface of a 3D free electron Topological insulator. The resulting insulating surface has $\sigma_{xy} = \pm 1/2(e^2/h)$ and $\kappa_{xy} = \pm 1/2$, since the domain wall carries a single chiral edge mode. (b) Interaction effects on the surface of a 3D Topological superconductor are irrelevant for weak interactions, given the linear dispersion of the Majorana cone surface states. However strong interactions can connect two surfaces without a phase transition - which implies the bulk phases are equivalent in the presence of interactions. Establishing this requires nonperturbative techniques.

Consider layers of 2D toric code models where just the e particle carries half charge. Now, in parallel to the constructions in lower dimensions, we consider a set of 3 layers, and form a bound state of $e_i m_{i+1} e_{i+2}$. This is a boson, which commutes with other triplets. For example $e_0 m_1 e_2$ and $e_1 m_2 e_3$ are mutual bosons. Also, it has integer charge and can be neutralized by a physical bosons. Hence condensing these triplets leads to a SRE 3D state, with all symmetries. However, it leaves behind an edge state - eg. e_0 is not confined. Similarly $m_0 e_1$ also commutes with the condensate. This is the new m particle of the toric code topological order, which is confined to the top layer. Note that it carries half charge, which is precisely what we wanted to construct. Here time reversal is explicitly preserved. By realizing this state on the surface of a 3D system ensures we never have to declare the edge physics (which would break time reversal symmetry).

This state corresponds to the surface of a 3D bosonic topological insulator (3D BTI), and models a surface, which is 'half' the 2D bosonic Integer Quantum Hall phase which has $\sigma_{xy} = Q^2/h$. Note, one can draw the following analogy to the free fermion topological insulator. A time reversal symmetry breaking perturbation can render the surface of the 3D TIU insulating. However, a domain wall between two opposite T-breaking domain on the surface necessarily has a single chiral mode along it (see Figure 1). Therefore the difference in Hall conductivity between the two domains is $\Delta\sigma_{xy} = 1(e^2/h)$. Also $\Delta\kappa_{xy} = 1$. By time reversal symmetry the two domains should have opposite Hall conductivities - hence we are forced to assign $\sigma_{xy} = +\frac{1}{2}\frac{e^2}{h}$ and $\kappa_{xy} = +\frac{1}{2}$ and the time reversed version to the other domain. Since a purely 2D free fermion system cannot have fractional Hall conductivities, it is not possible to screen this with a 2D layer. In a similar way we can build a 3D topological phase from the 2D integer Quantum Hall state of bosons, by including time reversal symmetry. This is the 3D BTI, whose surface state is described above.

In a very similar fashion one can model a state with a surface that is 'half' the chiral E_8 state, but time reversal symmetric when realized in 3D. This is the 3D bosonic topological superconductor (3D BTSc), and although is a symmetry protected topological phase, is not captured by the 'cohomology' approach of Chen, Liu, Gu and Wen. The surface topological order is the fermionic variant of the toric code - it has three nontrivial particles that have mutual π statistics, like the toric code, but all three particles have fermionic statistics. At first sight it might appear that this state is time reversal symmetric - but in fact it must carry chiral edge modes if realized in 2D. An explicit Chern Simons representation of this state is provided through the K matrix:

$$K = \begin{bmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{bmatrix} \tag{4}$$

The eigenvalues of this matrix are all the same sign, implying that there are 4 chiral modes if this state is realized in 2D, and hence always breaks $\mathcal{T}$ symmetry. However it may be realized on the surface of a 3D topological state with $\mathcal{T}$. Again this may be obtained via a coupled layer construction. A different approach [2] to realizing this phase is via an exactly soluble model, based on the following observation. It is well known in the context of the 2D Fractional Quantum Hall effect, that ground state wavefunctions can be related to correlation functions of the edge conformal field theory. The two coordinates of particles in the wave function are traded for a single spatial coordinate at the edge, and time. Can a similar approach be taken for 3D topological phases? While the obvious generalization is to relate the wave function written in terms of particle coordinates, a useful generalization is obtained by representing the wave function in terms of loops. Now, the amplitude of a particular loop configuration $\mathcal{C}$ in 3D space, $\Psi(\mathcal{C})$, is related to the space-time amplitude for a process in which the loops are imagined as world lines of particles in the surface topological order. Hence the loops come in different 'colors' corresponding to the nontrivial particles in the theory, and rules concerning how they fuse together etc. are determined by the topological data of the surface theory. For example, in the case of the 3-fermion topological order, the amplitude is

$$\Psi(\mathcal{C}) = \int Da\, e^{i\oint_{\mathcal{C}} j^\mu_I a^I_\mu} e^{iK_{IJ}\int \epsilon^{\mu\nu\lambda} a^I_\mu \partial_\nu a^J_\lambda} \tag{5}$$

[2]See F. Burnell et al., arXiv:1302.7072 for details

where the j define the loop structure. This can be converted into a exactly soluble model (Walker Wang model) on the cubic lattice .

1.1.2 Surface topological order of fermionic topological insulators and superconductors.

The well known fermionic Z_2 topological insulator is usually associated with a single dirac cone surface state. Breaking time reversal symmetry at the surface (eg. by introducing magnetic moments that order) can open a surface gap and render it insulating. However, this is not the only way to obtain an insulating surface - one can preserve all symmetries and obtain an insulator with topological order as for the bosonic SRE phases. Note, by the same logic as for the 3D BTI and 3D BTSc, the topological order is such that when realized in purely 3D it breaks $\mathcal{T}$ symmetry, and has $\sigma_{xy} = \frac{1}{2}e^2/h$ and $\kappa_{xy} = \frac{1}{2}$. That is - it is a candidate for a fractional Quantum Hall effect of electrons in a half filled Landau level. The most famous such candidate is the Moore-Read Pfaffian state, which may be thought of as Ising $\times\, U(1)_8$. More physically, one can imagine beginning with a superconductor of electrons, in a $p_x + ip_y$ state, where the Cooper pairs are effectively at $\nu_{Cooper} = 1/8$ filling [3] When the Cooper pairs form a bosonic Laughlin state, the Moore-Read state results. Unfortunately, while this state has the right σ_{xy} it has $\kappa_{xy} = 3/2$. Moreover, a quick glance at the topological spins of the quasiparticles reveals that it cannot be made time reversal symmetric even on the surface of a 3D topological insulator. Fortunately a simple variant is much more promising - one considers $p_x - ip_y$ superconductor in conjunction with the same Cooper pair Laughlin state, i.e. Ising$^* \times U(1)_8$. This state, dubbed the T-Pfaffian can be made time reversal symmetric on the surface of a 3D TI, but of course breaks it in 2D since it has a finite Hall conductance. A different but equivalent solution features the Moore-Read state in conjunction with a neutral anti-semion theory $U(1)_{-2}$. The surface topological order helps to understand how this classification is augmented in the presence of strong interactions wherein the electrons may pair to form bosons that exhibit a topological phase. The electrons could form Cooper pairs that then go into a 3D BTI phase. Or, the electrons could combine into neutral bosons that then enter a 3D BTSc phase. Both of these extend the original Z_2 Classification by an additional factor of Z_2. Wang, Potter and Senthil showed that this exhausts the set of 3D topological phases of interacting electrons with charge conservation and $\mathcal{T}$ symmetry.

The topological superconductors in 3D are protected by $\mathcal{T}$, and, for the physical case of $\mathcal{T}^2 = -1$ when acting on fermions, gives rise to an integer set of topological phases. One may imagine that combining pairs of electrons into neutral bosons, one can augment this classification by Z_2, by including the 3D BTSc in this list. However, it turns out that this phase is already present in the free fermion classification and corresponds to number $\nu = 8$ of the $\mathcal{Z}$ classification. Hence, there is no new phase. On the other hand, since this

[3] $\nu_{Cooper} = \frac{1}{4}\nu_{electron}$, since there are half as many Cooper pairs as electrons, and the magnetic field measured in units of the new flux quantum $h/2e$ is twice as large.

Symmetries	Free Fermions	Interacting Bosons	Interacting Fermions
U(1) (charge) and $\mathcal{T}$ Topological Insulator	Z_2 Class AII	Z_2^3	Z_2^3
$\mathcal{T}$ $\mathcal{T}^2 = (-1)^{N_F}$ TSc	Z Class DIII	Z_2^2	Z_{16}

Table 1: Topological phases in 3D with short range entanglement. The physically most relevant symmetries, corresponding to the topological insulator and superconductor.

phase has a Z_2 classification, this implies that two copies of $\nu = 8$, i.e. $\nu = 16$ is trivial. Therefore the interacting topological superconductor classification is reduced from the free fermion one $\mathcal{Z} \to \mathcal{Z}_{16}$. This observation is interesting since it represents a non-perturbative result in 2+1 dimensions. The surface of a topological superconductor with $\nu = 1$ has a Majorana cone with low energy dispersion:

$$H = -i\chi^T \left(\sigma^x \partial_x + \sigma^z \partial_y\right) \chi \tag{6}$$

where $\chi^T = (\chi_1, \chi_2)$. The surface with index ν then has ν flavors with the dispersion above. It is readily verified that weak interactions at the surface are irrelevant. From Eqn. 6, requiring that the action corresponding to the kinetic term is dimensionless (and both time and space have same dimensions $[t] = [x] \sim L$ the scaling dimension of the χ fields are $[\chi] \sim \frac{1}{L}$. The interaction term, written schematically as $\mathcal{S}_{int} \sim \int d^2 dt\, (\chi_a^T \sigma_y \chi_a)(\chi_b^T \sigma_y \chi_b)$ then has scaling dimension $[\mathcal{S}_{int}] \sim \frac{1}{L}$ which means that it is irrelevant at long scales. Therefore, the way $\nu = 16$ is connected to $\nu = 0$ is via strong interactions as shown in the figure 1b. Establishing this therefore requires a nonperturbative analysis. While bosonization provides such a tool for a 1+1D edge, establishing this for 2+1D requires new nonperturbative tools - such as working with the surface topological order or a dual vortex theory.