Duality in generalized Ising models

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Abstract This paper rests to a large extend on a paper I wrote quite awhile ago on *Duality in generalized Ising models and phase transitions without local order parameter*. It deals with Ising models with interactions containing products of more than two spins. In contrast to this old paper I will first give examples before I come to the general statements. Finally I will shortly mention a few, but very important developments, which have some relation to this paper, (i) the basic idea by Ken Wilson of a theory for quarks and gluons, and (ii) the idea by Thomas Weiland to discretize Maxwell's equations on lattices used here.

1 Introduction

In this contribution I consider a number of Ising models, which arose out of the question, whether there is duality for Ising models in dimensions larger than two. Indeed the idea of duality can be used to construct a whole class of such systems, which however, differ from conventional Ising models in some properties. First these models contain products of more than two Ising spins, Secondly they have no longer local order parameters, but they can still have two phases. For a number of these systems the order appears in the expectation value of the product of the spins along a loop, called Wilson-loop. It shows in the limit of large loops an area law at high temperatures and a perimeter law at low temperatures.

Such models, where the ising spins are replaced by members of groups, typically by the groups U, SU(2) and SU(3) have become important in high-energy physics in the description of quarks and gluons.

In sect. 2 I review the Kramers-Wannier duality for two-dimensional Ising models. In sect. 3 I introduce the model dual to the conventional three-dimensional Ising model. In sect. 4 the general concept of Ising models and

duality is introduced. In sect. 5 this is applied to general lattices and in sect. 6 to models on hypercubic lattices. The correlation functions are considered in sect. 7. The basic idea of lattice gauge theory is given in sect. 8 and a useful lattice for the discretization of Maxwell's equations is mentioned in sect. 9.

2 Kramers-Wannier Duality

Kramers and Wannier[6, 9] predicted in 1941 the exact critical temperature of the two-dimensional Ising model on a square lattice. They did this by comparing the high- and the low-temperature expansion for the partition function of this model. Consider a square lattice with $N_s = N_1 \times N_2$ lattice points and periodic boundary conditions. There is an Ising spin $S_{i,j} = \pm 1$ at each lattice site. The Hamiltonian reads

$$H = -J \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} (S_{i,j} S_{i,j+1} + S_{i,j} S_{i+1,j}). \tag{1}$$

High temperature expansion (*HTE*) We may rewrite

$$\begin{aligned} \mathrm{e}^{-\beta H} &= \prod_{i,j} (\cosh K + \sinh K S_{i,j} S_{i,j+1})(\cosh K + \sinh K S_{i,j} S_{i+1,j}) \\ &= (\cosh K)^{N_b} \prod_{i,j} (1 + \tanh K S_{i,j} S_{i,j+1})(1 + \tanh K S_{i,j} S_{i+1,j}) \end{aligned} \tag{2}$$

In order to determine the partition function, we may expand this expression in powers of $\tanh KSS'$ and sum over all spin configurations. This summation yields zero unless all spins appear with even powers. In this latter case the sum is 2^{N_s}. This is the case, when the interaction bonds form closed loops. That is at each lattice site meet an even number of bonds as shown in the upper figs. of fig. 1.

The partition function can be expanded

$$Z(K) = (\cosh K)^{N_b} f(\tanh K), \tag{3}$$

$$f(a) = \sum_l c_l a^l, \tag{4}$$

where $K = \beta J$. The coefficients c_l count the number of closed loops of length l, $c_0 = 1$, $c_2 = 0$, $c_4 = N_s$, $c_6 = 2N_s$, $c_8 = N_s(N_s + 9)/2$, etc. and $c_l = 0$ for odd l.

Low temperature expansion (*LTE*) We now consider the low temperature expansion on the dual lattice. The dual lattice is obtained by placing a spin $S^*(r^*)$ inside each of the squares (in general polygons) of the original lattice. We muliply spins S^* in polygons with a common edge and sum over these products, which in the case of the square lattice writes

$$H^* = -J^* \sum_{i,j} (S^*_{i-1/2,j-1/2} S^*_{i-1/2,j+1/2} + S^*_{i-1/2,j-1/2} S^*_{i+1/2,j-1/2}). \tag{5}$$

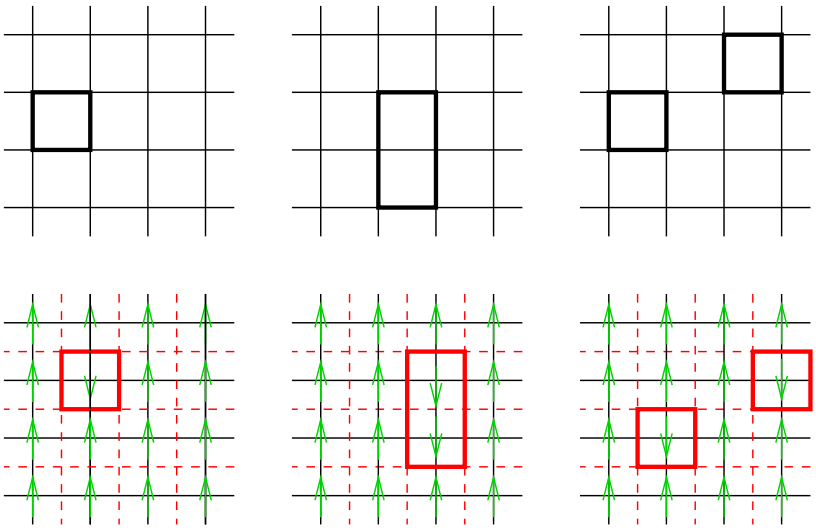

Figure 1: Examples for closed loops in the *HTE* and Bloch walls in the *LTE* on the dual lattice

Assuming positive J^* the states lowest in energy are those where all S^* are equal. There energy is

$$E^*_{min} = -N_b J^*, \tag{6}$$

where $N_b = 2N^*_s$ is the number of bonds.

Excited states are found by flipping some spins. Flipping one spin over costs an excitation energy $2lJ$, if the spin interacts with l other spins. Quite general the excitation energy is given by $2lJ$, if the flipped spins are surrounded by Bloch walls of a total number of l edges. In the case of the square lattice one obtains

$$Z^*(K^*) = 2\mathrm{e}^{N_b K^*} f(\mathrm{e}^{-2K^*}) \tag{7}$$

with f defined in (4).

Comparison Kramers and Wannier argued: If the partition function or equivalently the free energy has a singularity at the critical point and no other singularity, then it must be determined by

$$\mathrm{e}^{-2K_c} = \tanh K_c, \tag{8}$$

which yields

$$K_c = \frac{1}{2}\ln(1+\sqrt{2}) = 0.4407, \tag{9}$$

which indeed turned out to be correct from Onsager's exact solution[7]. Thus there is a relation between the partition function and similarly the free energy at high ($K < K_c$) and low ($K^* > K_c$) temperatures for

$$\tanh K = \mathrm{e}^{-2K^*} \quad \leftrightarrow \quad \tanh K^* = \mathrm{e}^{-2K} \quad \rightarrow \quad \sinh(2K)\sinh(2K^*) = 1. \tag{10}$$

The square lattice is called self-dual, since the *HTE* and the *LTE* are performed on the same lattice. This is different for the triangular lattice, where the *HTE* is performed on the triangular lattice and the *LTE* on the honeycomb lattice. See fig. 2. Then however the *HTE* of the triangular lattice and the *LTE* of the honeycomb lattice are given by the same sum $f(a)$,

$$Z_3^{\rm hte}(K) = 2^{N_{s3}}(\cosh K)^{N_b} f_3(\tanh K), \qquad Z_6^{\rm lte}(K) = 2{\rm e}^{N_b K} f_3({\rm e}^{-2K}), \quad (11)$$
$$Z_6^{\rm hte}(K) = 2^{N_{s6}}(\cosh K)^{N_b} f_6(\tanh K), \qquad Z_3^{\rm lte}(K) = 2{\rm e}^{N_b K} f_6({\rm e}^{-2K}), \quad (12)$$

where the number N_b of bonds are equal in both lattices and $N_{s3} = N_b/3$ and $N_{s6} = 2N_b/3$. The coefficients c_l in f_3 and f_6 count the number of closed loops on the triangular and the honeycomb lattice, resp.

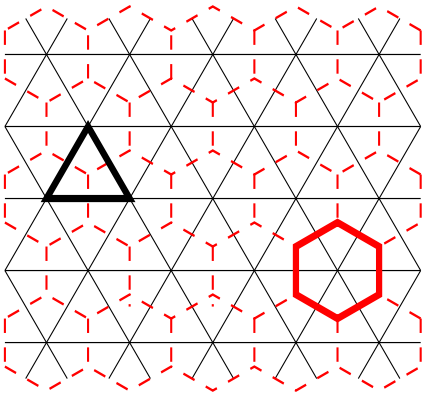

Figure 2: Triangular and dual hexagonal lattice. The thick black triangle indicates a product of 3 interactions on the triangular lattice contributing to *HTE* and the Bloch wall for a flipped spin on the hexagonal lattice. Similarly the thick red hexagon indicates a product of 6 interactions on the hexagonal lattice contributiong to *HTE* and the Bloch wall of a flipped spin on the triangular lattice.

As a consequence the partition functions $Z_3(K)$ and $Z_6(K^*)$ are directly related for K and K^* given by (10). One cannot directly read of the critical values K_c for these lattices. However, the Ising model on the honeycomb lattice can be related to that on the triangular lattice by means of the star-triangle transformation[9]. To do this one eliminates every other spin of the hexagonal lattice by summing $\sum_{S_0} {\rm e}^{KS_0(S_1+S_2+S_3)}$ one obtains $C{\rm e}^{K'(S_1S_2+S_1S_3+S_2S_3)}$, which yields the Ising model on the triangular lattice.

3 Duality in 3 dimensions

The basic question I asked myself, when I started my paper[10] on duality in generalized Ising models was: *Does there exist a dual model to the three-*

dimensional Ising model? It turned out, that there is such a model, but of a different kind of interaction. (Compare also [1])

In order to see this, I consider the low-temperature expansion of the 3d-Ising model on a cubic lattice. I start out from the ordered state and then flip single spins. These single spins are surrounded by closed Bloch walls. The expansion of the partition function is again of the form (4), but now with $c_2 = 0$, $c_4 = 0$, $c_6 = N_s$, $c_8 = 0$, $c_{10} = 3N_s$, $c_{12} = N_s(N_s - 7)/2$, etc.

The *HTE* of the dual model must be given by an interaction such that only closed surfaces yield a contribution. Thus locate a spin at each edge and introduce the interaction as a product of the spins surrounding an elementary square called plaquette. Thus the interaction of the dual model reads

$$\begin{aligned} H = -J \sum_{i,j,k} \quad (\quad & S_{i+1/2,j,k+1/2} S_{i+1/2,j+1/2,k} S_{i+1/2,j,k-1/2} S_{i+1/2,j-1/2,k} \\ + \quad & S_{i+1/2,j+1/2,k} S_{i,j+1/2,k+1/2} S_{i-1/2,j+1/2,k} S_{i,j+1/2,k-1/2} \\ + \quad & S_{i+1/2,j,k+1/2} S_{i,j+1/2,k+1/2} S_{i-1/2,j,k+1/2} S_{i,j-1/2,k+1/2}) \end{aligned} \tag{13}$$

It is the sum over three differently oriented plaquettes. They are shown in fig. 3.

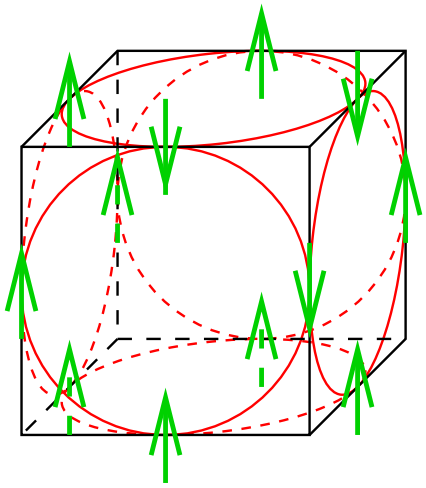

Figure 3: Elementary cube with spins. The red circles (ellipses) indicate, which four spins are multiplied in the interaction

S-independent products of $R(b)$ From fig. 3 it is obvious that the product of the six $R(b)$ around the cube does not depend on the spin configuration, since each spin appears twice in the product.

Gauge invariance This model has a local gauge invariance. Flipping all spins around the corner of a cube does not change the energy of the configuration. As an example in fig. 3 the three spins around the corner close to the center are flipped from the state, in which all spins are aligned upwards.

4 General Ising models and duality

4.1 General Ising models

We consider models with N_s Ising spins on lattice sites r described by a Hamiltonian

$$\beta H = -\sum_b K(b)R(b), \quad R(b) = \prod_r S(r)^{\theta(r,b)}, \quad \theta(r,b) \in \mathrm{Z}_{2+} \tag{14}$$

We call the b bonds; there number be N_b. $\theta(r,b)$ assumes the values 0 and 1. Obviously $R(b)$ is the product of the Ising spins $S(r)$ with $\theta(r,b) = 1$. We denote the rank of the matrix θ modulo 2 by N_θ. Thus there is at least one $\theta \times \theta$-submatrix, whoose determinant equals one modulo 2, whereas all $(\theta+1) \times (\theta+1)$-submatrices equal zero modulo 2. If we write

$$S(r) = (-)^{\sigma(r)}, \quad R(b) = (-)^{\rho(b)}, \quad \sigma(r), \rho(b) \in \mathrm{Z}_{2+} \tag{15}$$

then

$$\rho(b) = \sum_r \theta(r,b)\sigma(r). \tag{16}$$

If all $K(b)$ are positive, then one ground state is given by $S(r) = +1$. In general there will be several ground states. They obey $\rho(b) \equiv 0 \bmod 2$ for all b. We denote the corresponding configurations by $\sigma_0(r)$,

$$\sum_r \theta(r,b)\sigma_0(r) \equiv 0 \bmod 2. \tag{17}$$

There are $N_s - N_\theta$ linearly independent solutions $\{\sigma_0\}$. which yield 2^{N_g} ground state configurations with

$$N_g = N_s - N_\theta. \tag{18}$$

4.2 Duality

Besides the Ising model described by the Hamiltonian (14) we consider a second Hamiltonian

$$\beta^* H^* = -\sum_b K^*(b)R^*(b), \quad R^*(b) = \prod_{r^*} S^*(r^*)^{\theta^*(r^*,b)}. \tag{19}$$

with N_s^* spins $S^*(r^*)$ on lattice sites r^*. The bonds b are common to both Hamiltonians. Similarly we introduce the rank N_θ^* and obtain the ground state degeneracy $2^{N_g^*}$ with $N_g^* = N_s^* - N_\theta^*$.

The two models are called dual to each other, if two conditions are fulfilled: (i) the closure condition

$$\sum_b \theta(r,b)\theta^*(r^*,b) \equiv 0 \bmod 2 \tag{20}$$

for all pairs r, r^*, and
(ii) the completeness relation $N_m = 0$, where

$$N_m := N_b - N_\theta - N_{\theta^*}. \tag{21}$$

If these two conditions are fulfilled, and $K(b)$ and $K^*(b)$ are connected by (10), then the partition functions of the two models are related by

$$Y\{K\} = Y^*\{K^*\}, \tag{22}$$

where

$$Y\{K\} = Z\{K\} 2^{-(N_s+N_g)/2} \prod_b (\cosh(2K(b))^{-1/2} \tag{23}$$

and similarly for $Y^*\{K^*\}$.

Derivation of (22, 23) The partition function $Z\{K\}$ can be written in *HTE*

$$\begin{aligned} Z\{K\} &= \sum_{\{S(r)\}} \mathrm{e}^{-\beta H} = \sum_{\{S(r)\}} \prod_b \mathrm{e}^{K(b)R(b)} \\ &= \prod_b \cosh K(b) \sum_{\{\phi(b)\}} \prod_b \tanh K(b)^{\phi_b} \sum_{\{S(r)\}} \prod_b R(b)^{\phi_b}, \end{aligned} \tag{24}$$

with $\phi(b) \in \mathrm{Z}_{2+}$ independent for all b. Since

$$\prod_b R(b)^{\phi_b} = \sum_r S(r)^{\sum_b \theta(r,b)\phi(b)}, \tag{25}$$

those $\phi(b) = \phi_0(b)$ contribute, which obey the set of homogeneous equations

$$\sum_b \theta(r,b)\phi_0(b) \equiv 0 \bmod 2 \tag{26}$$

for all r. There are $N_b - N_\theta$ linearly independent solutions. Such there are in total $2^{N_b - N_\theta}$ solutions $\{\phi_0\}$. They contribute with the factor 2^{N_s}. Thus

$$Z\{K\} = 2^{N_s} \prod_b \cosh K(b) \sum_{\{\phi_0(b)\}} \prod_b \tanh K(b)^{\phi_0(b)}. \tag{27}$$

The partition function $Z^*\{K^*\}$ reads in the low-temperature expansion

$$\begin{aligned} Z^*\{K^*\} &= \sum_{\{S^*(r^*)\}} \mathrm{e}^{-\beta^* H^*} = N_g^* \sum_{\mathrm{closed}\{b\}} \prod_b \mathrm{e}^{K^*(b)R^*(b)} \\ &= N_g^* \prod_b \mathrm{e}^{K^*(b)} \sum_{\mathrm{closed}\{b\}} \prod_b \mathrm{e}^{-2K^*(b)\rho^*(b)}, \end{aligned} \tag{28}$$

since $R^*(b) = 1 - 2\rho^*(b)$. The closed sets of $R(b)$ appearing in the sum obey the homogeneous equations

$$\sum_b \theta(r,b)\rho^*(b) \equiv 0 \bmod 2 \tag{29}$$

The closure condition yields

$$\sum_b \theta(r,b)\rho^*(b) = \sum_b \sum_{r^*} \theta(r,b)\theta^*(r^*,b)\sigma(r^*) \equiv 0 \bmod 2. \tag{30}$$

Therefore $\rho^*(b)$ obeys eq. (26) and the partition function may be written

$$Z^*\{K^*\} = \prod_b \mathrm{e}^{K^*(b)} \sum_{\{\phi_0\}} N\{\phi_0\} \prod_b \mathrm{e}^{-2K^*(b)\phi_0(b)}, \tag{31}$$

where $N\{\phi_0\}$ is the number of configurations $\{\sigma^*\}$, which yields $\{\phi_0\}$

$$\phi_0(b) = \sum_{r^*} \theta^*(r^*,b)\sigma^*(r^*) \equiv 0 \bmod 2. \tag{32}$$

If it has no solution, then $N\{\phi_0\} = 0$, otherwise it has $N\{\phi_0\} = 2^{N_g^*}$ solutions. For $\beta^* = 0$ one has

$$Z^* = 2^{N_s^*} = \sum_{\{\phi_0\}} N\{\phi_0\}, \tag{33}$$

which implies that the number of sets $\{\phi_0\}$ yielding $N\{\phi_0\} = 2^{N_g^*}$ is $2^{N_s^*-N_g^*} = 2^{N_\theta^*}$.

If the completeness condition is fulfilled, then all sets $\{\phi_0\}$ obeying (26) are contained in $Z^*\{K^*\}$. Thus

$$Z^*\{K^*\} = 2^{N_g^*} \prod_b \mathrm{e}^{K^*(b)} \sum_{\{\phi_0\}} \prod_b \mathrm{e}^{-2K^*(b)\phi_0(b)}. \tag{34}$$

The sums over $\{\phi_0\}$ are the same for $Z^{\{}K\}$ and $Z^*\{K^*\}$ in (27) and (34). Denoting

$$f\{a\} := \sum_{\{\phi_0\}} \prod_b a(b)^{\phi_0(b)}, \quad C := \prod_b \frac{\cosh K(b)}{(\cosh(2K(b)))^{1/2}} \tag{35}$$

one obtains

$$\prod_b \frac{\mathrm{e}^{K^*(b)}}{(\cosh(2K^*(b)))^{1/2}} = 2^{N_b} C \tag{36}$$

and

$$Y\{K\} = 2^{(N_s-N_g)/2} C f(\tanh K), \tag{37}$$

$$Y^*\{K^*\} = 2^{(N_b+N_g^*-N_s^*)/2} C f(\mathrm{e}^{-2K^*}) = 2^{(N_s-N_g+N_m)/2} C f(\mathrm{e}^{-2K^*}). \tag{38}$$

This yields the duality relation (22) for $N_m = 0$.

If $N_m > 0$, then the summation in (34) does not extend over the full set $\{\phi_0\}$. Let the sum denote by $f'\{a\}$ instead of $f\{a\}$ and

$$Y^*\{K^*\} = 2^{(N_s-N_g+N_m)/2} C f'(\mathrm{e}^{-2K^*}). \tag{39}$$

Since all terms in the sum f are positive, one obtains $f' < f$ and thus the inequality

$$Y^*\{K^*\} < 2^{N_m/2} Y\{K\}. \tag{40}$$

We have obtained this relation from the *HTE* of Z and the *LTE* of Z^*. If instead we consider the *HTE* of Z^* and the *LTE* of Z, then we obtain a second inequality, in total

$$2^{-N_m/2} Y\{K\} < Y^*\{K^*\} < 2^{N_m/2} Y\{K\}. \tag{41}$$

The difference in the free energy per polytope due to the factors $N_m^{\pm 1/2}$ in (41) vanishes in the thermodynamic limit, if N_m does not increase in the thermodynamic limit.

Example: Two-dimensional Ising model The two-dimensional Ising model with N_s spins on the square-lattice yields $N_s^* = N_s$, $N_b = 2N_s$, $N_g = N_g^* = 1$, and thus $N_m = 2$. The closed loops which show up in the *HTE*, but not in the l.t.e. are those, where one loop runs around the torus in one or the other or both directions. This corresponds to antiperiodic boundary conditions. Denoting the partition function with boundary conditions $S_{i,j} = s_x S_{i+N_1,j} = s_y S_{i,j+N_2}$ by Z_{s_x,s_y} one obtains the exact relation

$$Y\{K\} = 1/2(Y_{++}\{K^*\} + Y_{+-}\{K^*\} + Y_{-+}\{K^*\} + Y_{--}\{K^*\}). \tag{42}$$

The difference in the free energy per polytope due to the factors $N_m^{\pm 1/2}$ in (41) vanishes in the thermodynamic limit.

5 Lattices and their dual lattices

The models considered up to now are generalized to models in arbitrary dimensions d. I divide the d-dimensional hypervolume into C_d d-polytopes $B^{(d)}$. These are bounded by $(d-1)$-polytopes $B^{(d-1)}$. Generally the k-polytopes $B^{(k)}$ are bounded by $(k-1)$-polytopes $B^{(k-1)}$. Their number is denoted by C_k. The 0-polytopes are simply the C_0 corners $B^{(0)}$ of the d-polytopes.

I associate lattice points $r^{(k)}$ lying in the k-polytopes $B^{(k)}$.

For given n a bond $b(r^{(n)})$ is associated to each $r^{(n)}$. C_{n-1} Ising spins $S(r^{n-1})$ are introduced and define $\theta(r^{(n-1)}, b(r^{(n)})) = 1$, if $B(r^{n-1})$ is part of the boundary of $B(r^{(n)}$. Otherwise $\theta(r^{(n-1)}, b(r^{(n)})) = 0$. Similarly C_{n+1} Ising spins $S^*(r^{(n+1)})$ and $\theta^*(r^{(n+1)}, b(r^{(n)}) = 1$, if $B(r^{(n)})$ is part of the boundary of $B(r^{(n+1)})$. We observe now that the closure condition (20) is fulfilled. If $B^{(n-1)}$ is an $(n-1)$ dimensional cell at the boundary of $B^{(n+1)}$, then $B^{(n-1)}$ is, where two boundaries $B^{(n)}$ of $B^{(n+1)}$ meet. Then the expression on the l.h.s. of (20) equals 2, otherwise 0. This defines together with couplings K and K^* models (14) and (19).

The dual lattice is obtained in the following way: The points $r^{(d)}$ are the corners of the dual lattice. Pairs of points $r^{(d)}$ are connected by 1-dimensional

cells $B^{*(1)}$, if the corresponding two cells are separated by a $B^{(d-1)}$. The 1-dimensional cells $B^{*(1)}$ crossing the cells $B^{(d-1)}$ around a given a cell $B^{(d-2)}$ is the boundary of a 2-dimensional cell $B^{*(2)}$. Generally the k-dimensional cells $B^{*(k)}$ crossing the cells $B^{(d-k)}$ around a cell $B^{(d-k-1)}$ is the boundary of a $(k+1)$-dimensional $B^{*(k+1)}$. It is reasonable to define the intersection of a cell $B^{*(k)}$ with its corresponding cell $B^{(d-k)}$ as the point $r^{*(k)} = r^{(d-k)}$. Thus the number of cells $B^{*(k)}$ and points $r^{*(k)}$ is $C^*_k = C_{d-k}$.

I define now

$$\theta(r^{(k+1)}, r^{(k)}) = \begin{cases} 1 & B^{(k)} \text{ on boundary of } B^{(k+1)}, \\ 0 & B^{(k)} \text{ not on boundary of } B^{(k+1)} \end{cases} \tag{43}$$

Closure relation: An important property of the lattices is the closure relation: Consider a pair $r^{(k+1)}$ and $r^{(k-1)}$. They lie in cells $B^{(k+1)}$ and B^{k-1}. Then

$$\sum_{r^{(k)}} \theta(r^{(k+1)}, r^{(k)})\theta(r^{(k)}, r^{(k-1)}) \equiv 0 \bmod 2. \tag{44}$$

Proof: If $B^{(k-1)}$ is on the boundary of $B^{(k+1)}$, then two cells $B^{(k)}$ on the boundary of $B^{(k+1)}$ have $B^{(k-1)}$ as boundaries. If $B^{(k-1)}$ is not at the boundary of $B^{(k+1)}$, then none of the $B^{(k)}$ on the boundary of $B^{(k+1)}$ has $B^{(k-1)}$ as boundary. This proofs (44).

5.1 Models on the lattice

The model M_{dn} have C_{n+1} spins on lattice sites $r^{(n-1)}$ with an interaction defined by the bonds

$$R(b) = \prod_{r^{(n-1)}} S_(r^{(n-1)})^{\theta(r^{(n-1)}, r^{(n)}(b))}. \tag{45}$$

The dual model $M^*_{d,d-n}$ has $C^*_{d-n+1} = C_{n-1}$ spins at lattice sites $r^{(n+1)}$,

$$R^*(b) = \prod_{r^{(n+1)}} S^*(r^{(n+1)})^{\theta(r^{(n)}(b), r^{(n+1)})}. \tag{46}$$

Gauge invariance Flipping all spins close to a point $r^{(n-2)}$,

$$S(r^{(n-1)}) \to (-)^{\theta(r^{(n-2)}, r^{n-1})} S(r^{(n-1}) \tag{47}$$

does not change the energy of the system, since any $R(b)$ is multiplied by

$$(-)^{\sum_{r^{n-1}} \theta(r^{(n-2)}, r^{(n-1)}))\theta(r^{(n-1)}, r^n(b))}, \tag{48}$$

which due to the closure relation (44) yields one.

Spin-independent products $R(b)$ The product over all $R(b)$ around a given r^{n+1}, that is

$$\prod_b R(b)^{\theta(r^n(b), r^{n+1})} = \prod_{r^{(n-1)}} S(r^{(n-1)})^{\sum_{r^{(n)}} \theta(r^{(n-1)}, r^{(n)})\theta(r^{(n)}, r^{(n+1)})} = 1. \tag{49}$$

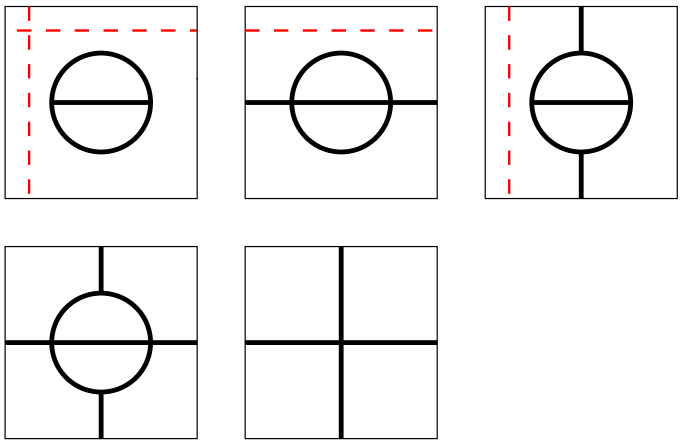

Figure 4: Example of two-dimensional lattices in a periodicity square. The upper three examples do not yield $\chi = 0$, since walls indicated in red lines do not intersect any edges. The lower two examples yield $\chi = 0$.

does not depend on the spin configuration, since due to the closure relation (44) it yields one.

Generalized Euler characteristic The well-known Euler characteristic in $d = 2$ dimensions

$$\chi = C_0 - C_1 + C_2, \tag{50}$$

where C_0 is the number of vertices (corners), C_1 number of edges, and C_2 the number of faces, depends only on the topology of the surface. For the plane one has $\chi = 2$, if the outer face is also counted. For the torus one has $\chi = 0$. This characteristic can be generalized to arbitrary dimension d,

$$\chi = \sum_{m=0}^{d} (-)^m C_m. \tag{51}$$

Any lattice with the same boundaries (topology) can be created from any other one by means of the following steps and their inverses:

Step: An m-dimensional polytope is divided into two such cells by creating an $(m+1)$-dimensional polytopes between them. Then both C_m and C_{m+1} increase by one and χ is conserved.

For periodic boundary conditions one obtains $\chi = 0$, since we may cut the lattice in one direction, double it and glue the two parts together. Then all C_m have doubled, but $\chi = 2\chi$ and thus vanishes. It is presumed that it is not possible to introduce an additional 'wall' $B^{(d-1)}$ in any periodic direction, which does not intersect any of the original cells $B^{(d-1)}$. See fig. 4.

Degeneracy We consider the change of N_g resulting from the application of a step as defined below (51): If $m > n$, then the Hamiltonian is unchanged.

For $m = n$ one bond and thus one interaction is duplicated without change of degeneracy. For $m = n-1$ one spin is duplicated, but for the ground state both must equal. For $m = n-2$ there is also one additional spin. Taking this spin aligned upwards, one obtains again the ground state. But changing the signs of all spins lying on bonds adjacent to one $B^{(n-1)}$ at the boundary of the new bond, one obtains another ground state. Then the system has twice the degeneracy of the original system. The hamiltonian does not chage for $m < n-2$. Therefore we obtain

$$N_g = t_g + \sum_{m=0}^{n-2} (-)^{n-m} C_m, \tag{52}$$

where t_g depends only on the boundary condition. Similarly one obtains

$$N_g^* = t_g^* + \sum_{m=n+2}^{d} (-)^{m-n} C_m. \tag{53}$$

Thus

$$\begin{aligned} N_m &= N_b - N_s + N_g - N_s^* + N_g^* = C_n - C_{n-1} - C_{n+1} + N_g + N_g^* \\ &= \sum_{m=0}^{d} (-)^{n-m} C_m + t_g + t_g^* = (-)^n \chi + t_g + t_g^* \end{aligned} \tag{54}$$

We argue after (57) that

$$t_g = \binom{d-1}{n-1}, \quad t_g^* = \binom{d-1}{d-n-1}, \quad N_m = \binom{d}{n}. \tag{55}$$

for periodic boundary conditions.

6 The models $M_{d,n}$ on hypercubic lattices

The models $M_{d,n}$ are defined on hypercubic lattices. The polytopes $B^{(k)}$ are k-dimensional hypercubes with edges of unit length. The lattice points $r^{(k)}$ have k integer coordinates and $r^{(d-k)}$ half integer coordinates, that is they are $1/2$ modulo 1. We assume periodic boundary conditions, then

$$C_k = \binom{d}{k} C_d. \tag{56}$$

The model M_{dn} has spins on sites $r^{(n-1)}$. The dual model $M_{d,d-n}^*$ has spins on sites $r^{(n+1)} = r^{*(d-n-1)}$. It is the model $M_{d,d-n}$ shifted by $1/2$ in all coordinates.

Due to the above conditions $r^{(n+1)}$ and $r^{(n-1)}$ can only have a bond in common, if they agree in $d-2$ coordinates and differ only in 2 coordinates. Let these coordinates be (i, j) and $i \pm 1/2, j \pm 1/2$. Then they have two bonds in common as claimed before: $(i, j \pm 1/2)$ and $(i \pm 1/2, j)$ and fulfill the closure condition.

Gauge invariance and degeneracy If $n > 1$, then one may turn over all spins closest to a given point $r^{(n-2)}$ without changing the energy of the system. Thus these systems have a local gauge invariance. This leads to a high degeneracy of the ground state. From (52) we obtain

$$N_g = t_g + \sum_{m=0}^{n-2} (-)^{n-m} \binom{d}{m} C_d = t_g + \binom{d-1}{n-2} C_d \tag{57}$$

For only one hypercube $C_d = 1$ in the periodic lattice one obtains $N_g = N_s$, since periodic boundary conditions require that the spins in the products $R(b)$ are pairwise equal, we obtain t_g as in (55). Thus

$$N_g = \binom{d-1}{n-1} + \binom{d-1}{n-2} C_d. \tag{58}$$

Similarly we obtain

$$N_g^* = \binom{d-1}{d-n-1} + \binom{d-1}{d-n-2} C_d. \tag{59}$$

and N_m given in (55).

Spin-independent products R(b) In the h.t.e. we sum over products of $R(b)$, which yield unity for arbitrary spin-configurations. A large number of them are obtained by taking the product of all $R(b(r^{(n)}))$, where the $r^{(n)}$ are neighbors of any given $r^{(n+1)}$, since $S(r^{(n-1)})$ appears twice in the product, if $r^{(n+1)}$ and $r^{(n-1)}$ are closest, otherwise $S(r^{(n-1)})$ does not appear. Obviously also the products of such 'elementary products' of $R(b)$ are spin-independent.

Self-duality Obviously the models M_{dn} on the hypercubic lattice is self-dual, if $d = 2n$. This is the case for $M_{2,1}$, which is the two-dimensional Ising model on the square lattice. But also the four-dimensional model $M_{4,2}$ with the plaquette interaction is self-dual. Both have the phase transition at $K_c = 0.4407$, (9). The Ising model $M_{2,1}$ shows a continuous transition. Creutz, Jacobs, and Rebbi[2] have investigated the model $M_{4,2}$ by Monte Carlo techniques. They determined $\langle R(b) \rangle$ as a function of K. They found a first order transition with hysteresis. By increasing K the system showed superheating until ≈ 0.48 and by decreasing K undercooling until ≈ 0.40. Starting from a mixed phase the phase transition was located between 0.43 and 0.45.

Duality can be generalized to Abelian groups $Z(N)$. Let $S(r) = \mathrm{e}^{2\pi \mathrm{i} p/N}$ with $p = 0, ...N-1$ and the energy assigned to two product of two spins in states p and p' by $E_{p-p'}$, then the weights $\omega_{p-p'} = \mathrm{e}^{-\beta E_{p-p'}}$ and their dual are related by the Fourier transform[11]

$$\omega_p^* = N^{-1/2} \sum_{p}{}' \mathrm{e}^{2\pi \mathrm{i} p p'/N}. \tag{60}$$

This can be generalized to the models M_{dn}. The models M_{42} are self-dual for Z_N with $N = 3, 4$ and the critical K_c are determined[5, 15]. Monte-Carlo calculations[3] confirm these transition temperatures for $N = 3, 4$. Corresponding calculations yield two phase transitions for $N \geq 5$.

7 Correlations

Non-vanishing correlations are only obtained for gauge-invariant products. These are products of $R(b)$. In particular we consider the product of spins on the boundary of an n-dimensional hypercube of M_{dn}. The *HTE* yields

$$\begin{aligned}\langle\prod_r S(r)\rangle &= (\tanh K + 2(d-n)(\tanh K)^{1+2n} + ...)^v, \quad n > 1, \qquad (61)\\ &= \frac{1}{2}[\tanh K + (2(d-1))^{1/2}(\tanh K)^2 + ...]^v \\ &+ \frac{1}{2}[\tanh K + (2(d-1))^{1/2}(\tanh K)^2 + ...]^v, \quad n = 1. \qquad (62)\end{aligned}$$

where v is the volume of the hypercube. For $n = 1$ this is the distance between the two spins; for $n = 2$ it is the area spanned by the spins. The *LTE* yields

$$\begin{aligned}\langle\prod_r S(r)\rangle &= (1 - \mathrm{e}^{4(d-n+1)K} + ...)^f, \quad n < d,\\ \langle\prod_r S(r)\rangle &= (1 - 2\mathrm{e}^{-2K} + ...)^v, \quad n = d, \qquad (63)\end{aligned}$$

where f is the hyperarea of the boundary of the hypercube (for $n = 1$ it is the number $f = 2$ of ends of the line; for $n = 2$, f is the perimeter of the square). Thus the behavior of the correlation functions of large hypercubes differs in the high and low temperature phases, and we expect

$$\langle\prod_r S(r)\rangle \propto \begin{cases} \mathrm{e}^{-v/v_0(T)} & T > T_c,\ n < d \\ \mathrm{e}^{-f/f_0(T)} & T < T_c,\ n < d \end{cases} \qquad (64)$$

We attribute the qualitatively different asymptotic behavior in both temperature regions to different states of the system above and below a critical temperature T_c.

The model $\mathbf{M_{dd}}$ The only restriction on the $R(b)$ is that the product of all of them equals one. Consequently the partition function reads

$$Z(K) = 2^{N_s}[(\cosh K)^{N_b} + (\sinh K)^{N_b}] \qquad (65)$$

The expectation value of a product of v factors R yields

$$\langle\prod_b R(b)\rangle = \frac{(\tanh K)^v + (\tanh K)^{N_b - v}}{1 + (\tanh K)^{N_b}}. \qquad (66)$$

The models M_{dd} do not show a phase transition. Among these models is M_{11}, a closed linear chain of Ising spins.

7.1 Dislocations

We consider systems with magnetic dislocations. Let the operator $M(b)$ change the sign of $K(b)$. We introduce $\phi^*(b) = 1$ for bonds with changed signs, and

$\phi^*(b) = 0$ for bonds with unchanged coupling. Then the expectation value of the product of $M(b)$s is

$$\begin{aligned}\langle\prod_b M(b)^{\phi^*(b)}\rangle &= \langle\prod_b \mathrm{e}^{-2\phi^*(b)K(b)R(b)}\rangle \\ &= \frac{Z\{(-)^{\phi^*}K\}}{Z\{K\}} = \frac{Y\{(-)^{\phi^*}K\}}{Y\{K\}}.\end{aligned} \tag{67}$$

From (10) we obtain $\tanh((-)^{\phi^*}K) = \mathrm{e}^{-2K^*-\mathrm{i}\pi\phi^*}$ and thus

$$\begin{aligned}\langle\prod_b M(b)^{\phi^*(b)}\rangle &= \frac{Y\{K^* + \mathrm{i}\pi\phi^*/2\}}{Y\{K^*\}} \\ = \quad \mathrm{i}^{-\sum_b \phi^*(b)}\langle\prod_b \mathrm{e}^{\mathrm{i}\pi\phi^*(b)R^*(b)/2}\rangle\{K^*\} &= \langle\prod_b R^*(b)^{\phi^*(b)}\rangle\{K^*\}.\end{aligned} \tag{68}$$

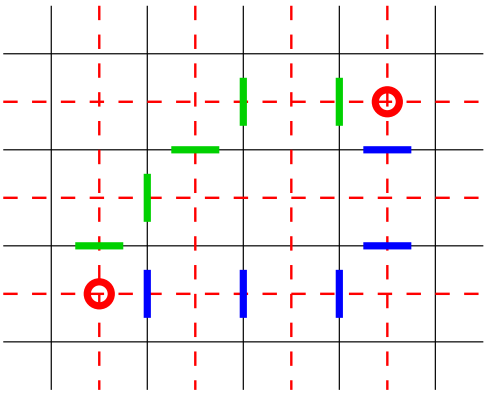

Figure 5: Correlation between the two spins at the dots

Interpretation Kadanoff and Ceva[4] introduced this concept for the two-dimensional Ising model $M_{2,1}$. Take a sequence of bonds b indicated by either the blue or the green bars between the two spins at the sites indicated in fig. 5 by two red circles, then $\langle\prod_b R^*(b)\rangle$ is the product of these two Ising spins at K^*. It equals the ratio of the partition functions with the changed bonds and the unchanged bonds, thus the exponential of the difference ΔF of the free energy without and with the changed bonds at K,

$$\langle S(r^*)S(r^{*'})\rangle(K^*) = \mathrm{e}^{-\Delta F(K)}. \tag{69}$$

If K is in the paramagnetic region, then the disturbance of the bonds yields a contribution to δF only close to the points, where this line of bonds ends. Thus for large separation of the two spins it approaches a finite value, which corresponds to the square of the magnetization at K^*. On the other hand if K

is in the ferromagnetic region, then the disturbance will change the free energy proportional to the distance between the two spins $S(r^*)$ and $S(r^{*'})$, which yields an exponential decay of the correlation function.

Let us now consider $M_{3,1}$ and $M_{3,2}$. Change the sign of the interaction $\sum_{ij} S_{ijk}S_{ijk+1}$ over a whole region (area) in the plane spanned by ij. Analogous to the two-dimensional Ising model, the change $\delta F(K)$ will be proportional to the perimeter f for paramagnetic K and proportional to the area v for ferromagnetic K. The product $\prod_b R^*(b)$ is now the product of the Ising spins along the perimeter of the dislocations. Consequently the expectation value decays proportional to $\mathrm{e}^{-f/f_0(T^*)}$ at low temperatures T^* and proportional to $\mathrm{e}^{-v/v_0(T^*)}$ at high temperatures T^* in accordance with (64).

Local order parameter If all states are taken into account. the correlations different from zero are only obtained from products of R, For $n = 1$ the product of two spins $S(0)S(r)$ can be written as product of Rs. For $n > 1$ products of spins $\prod_k S(a_k) \prod_l S(r + a_l)$ with with a_k and a_l restricted to some finite region $|a_k| < c$, $|a_l| < c$ yield only finite correlations for distances $r > 2c$, if both $\prod_k S(a_k)$ and $\prod_l S(r + a_l)$ are separately gauge invariant, that is, they are expressed as finite products of R. However, with (67, dis2) expectations of products of R in one phase can be expressed by correlations in the other phase

$$\langle \prod_{\text{some } b} R(b)\rangle\{K\} = \langle \prod_{\text{same } b} (\cosh(2K^*(b)) - R^*(b)\sinh(2K^*(b))) \tag{70}$$

Thus since there is no long range order in the high temperature phase, there can be none in the low temperature phase,

$$\lim_{r\to\infty} (\langle \prod_k S(a_k) \prod_l S(r + a_l)\rangle - \langle \prod_k S(a_k)\rangle\langle \prod_l S(a_l)\rangle) = 0. \tag{71}$$

Thus there is no local order parameter for models M_{dn} with $n > 1$. This argument does not apply for $n = 1$, since in this case the number of Rs in the product increases with $|r|$.

8 Lattice gauge theories

We have seen that models M_{dn} with $n > 1$ show local gauge invariance. Such models are related to quantum chromodynamics. The basic idea first formulated by Wilson[13] is to start from the lattice, we introduced as M_{42}. (For a retrospect by Wilson see [14]. Many reprints on this subject are compiled in Rebbi's book[8]). The degrees of freedom are now denoted by U in place of S. These U are elements of a group. It may be a finite or a continuous group, it may be an Abelian or non-Abelian group. In the case of QCD one considers the group SU(3). Let us denote the U placed on the link between lattice sites i and j by U_{ij}, where one requires $U_{ji} = U_{ij}^{-1}$. The action is a sum of terms

$$g^{-2} \sum_{\text{plaquettes}} (1 - \frac{1}{N}\Re\mathrm{tr}(U_{ij}U_{jk}U_{kl}U_{li})), \tag{72}$$

where N is the dimension of U. In addition one introduces quarks (fermions) with interaction

$$g'^{-2} \sum_{\text{links}} \psi_i^\dagger U_{ij} \psi_j. \tag{73}$$

These interaction terms are invariant under local gauge transformations

$$\psi_j \to G_j \psi_j, \quad \psi_j^\dagger \to \psi_j^\dagger G_j^\dagger, \quad U_{ij} \to G_i U_{ij} G_j^\dagger. \tag{74}$$

The couplings depend on temperature and pressure of the hadron system. At low temperature and pressure the correlations fall of with an area law. Since the action is an integral over time, this behaviour corresponds to an increase of the effective potential between quarks proportional to the distance between them. The gradient of the potential is called string tension and given by $1/\upsilon_0(T)$ in (64). This potential binds three quarks, which constitute a hadron. Or one quark and one antiquark are bound and constitute a meson. Generally the difference between the number of quarks and antiquarks has to be a multiple of three. At high temperature and high pressure the system forms a quark-gluon plasma. This corresponds to the phase in which the correlation increases proportional to the perimeter of the loop. Then the effective potential between the quarks stays finite at large distances and the quarks are rather free to move in this plasma.

9 Electromagnetic field

The electromagnetic field in QED and its coupling to charged particles can be described similarly with the group $U(1)$,

$$U_{ij} = \mathrm{e}^{\mathrm{i} \int_j^i A_\mu \mathrm{d}x^\mu} \tag{75}$$

Then

$$\begin{aligned} &\mathrm{tr}(U_{r,r+a^\mu e_\mu} U_{r+a^\mu e_\mu, r+a^\mu e_\mu + a^\nu e_\nu} U_{r+a^\mu e_\mu + a^\nu e_\nu, r+a^\nu e_\nu} U_{r+a^\nu e_\nu}) \\ &\approx \mathrm{e}^{\mathrm{i} a^\mu a^\nu F_{\mu\nu}(r + (a^\mu e_\mu + a^\nu e_\nu)/2)} \end{aligned} \tag{76}$$

with the electromagnetic field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{77}$$

Since only the real part of $\mathrm{tr}(\prod U)$ contributes, one obtains in leading order the well-known action of the electromagnetic field proportional to $F_{\mu\nu} F^{\mu\nu}$. If one performs the continuum limit ($a \to 0$) then only these terms survive.

The discretized Maxwell equations can be solved on such a lattice[12]. One places the components A_μ on sites $r^{(1)}$, the six electromagnetic field components $F_{\mu\nu}$ on sites $r^{(2)}$, the components of the charge and current densities on sites $r^{(1)}$. Lorenz gauge and charge conservation can be put on sites $r^{(0)}$.

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